04-.05-2019 MECE 548 Midterm

(Q1) $\dot{x} = \begin{bmatrix} 2 & 0 \\ t & 2 \end{bmatrix} x$ What is $\phi(t, t_0)$

$\dot{x}_1 = 2x_1 \longrightarrow x_1 = x_1(0)e^{2t}$

$\dot{x}_2 = tx_1 + 2x_2 \longrightarrow \dot{x}_2 = t\left[x_1(0)e^{2t}\right] + 2x_2$

$*$ $\dot{x}_2 - 2x_2 = tx_1(0)e^{2t}$

$x_{2h} = Ke^{2t}$ homogeneous solution

$x_{2p} = Nt^2e^{2t}$ (particular solution)

$\dot{x}_{2p} = 2Nte^{2t} + Nt^2 2e^{2t}$ (put in $*$)

$2Nte^{2t} + Nt^2 2e^{2t} - 2\left(Nt^2e^{2t}\right) = tx_1(0)e^{2t}$

$2Nte^{2t} = tx_1(0)e^{2t}$ $N = \frac{x_1(0)}{2}$

$x_{2p} = \frac{x_1(0)}{2}t^2e^{2t}$ // $\rightarrow$ particular solution

$x_2 = x_{2p} + x_{2h} = \frac{x_1(0)}{2}t^2e^{2t} + Ke^{2t}$

$x_2(0) = K$ //

$x_1 = x_1(0)e^{2t}$

$x_2 = \frac{x_1(0)}{2}t^2e^{2t} + x_2(0)e^{2t}$

Then

$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ \frac{1}{2}te^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$\phi(t, 0) = \begin{bmatrix} e^{2t} & 0 \\ \frac{1}{2}t^2e^{2t} & e^{2t} \end{bmatrix}$ //

$$\phi(t,0) = \begin{pmatrix} e^{2t} & 0 \\ \frac{1}{2}t^2 e^{2t} & e^{2t} \end{pmatrix}$$

$$\phi(0,t) = \phi^{-1}(t,0) = \frac{\begin{bmatrix} e^{2t} & 0 \\ -\frac{1}{2}t^2 e^{2t} & e^{2t} \end{bmatrix}}{e^{2t}\cdot e^{2t} - \left(0 \cdot \frac{1}{2}t^2 e^{2t}\right)}$$

$$\phi(0,t) = \begin{bmatrix} e^{-2t} & 0 \\ -\frac{1}{2}t^2 e^{-2t} & e^{-2t} \end{bmatrix}$$

$$\phi(0,t_0) = \begin{pmatrix} e^{-2t_0} & 0 \\ -\frac{1}{2}t_0^2 e^{-2t_0} & e^{-2t_0} \end{pmatrix}$$

$$\phi(t,t_0) = \phi(t,0)\cdot\phi(0,t_0) = \begin{pmatrix} e^{2t} & 0 \\ \frac{1}{2}t^2 e^{2t} & e^{2t} \end{pmatrix}\begin{bmatrix} e^{-2t_0} & 0 \\ -\frac{1}{2}t_0^2 e^{-2t_0} & e^{-2t_0} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2(t-t_0)} & 0 \\ \frac{1}{2}t^2 e^{2(t-t_0)} - \frac{1}{2}t_0^2 e^{2(t-t_0)} & e^{2(t-t_0)} \end{bmatrix}$$

$$\phi(t,t_0) = \begin{bmatrix} e^{2(t-t_0)} & 0 \\ \frac{1}{2}(t^2-t_0^2)e^{2(t-t_0)} & e^{2(t-t_0)} \end{bmatrix}$$

validation

$$\phi(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\dot\phi(t,0) = \begin{bmatrix} 2e^{2t} & 0 \\ te^{2t}+t^2e^{2t} & 2e^{2t} \end{bmatrix} \checkmark$$

$$= A\,\phi(t,0)$$

$$= \begin{pmatrix} 2 & 0 \\ t & 2 \end{pmatrix}\begin{bmatrix} e^{2t} & 0 \\ \frac{1}{2}t^2e^{2t} & e^{2t} \end{bmatrix}$$

$$\dot\phi(t,0) = \begin{pmatrix} 2e^{2t} & 0 \\ te^{2t}+t^2e^{2t} & 2e^{2t} \end{pmatrix}$$

Q2 (a)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.25 & 0.25 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = [1 \quad 1 \quad 0] x$$

in controllable canonical form

$$H_1(s) = \frac{s+1}{s^3 + s^2 - 0.25s - 0.25} = \frac{(s+1)}{(s+1)(s^2 - 0.25)}$$

$$= \frac{1}{s^2 - 0.25} \checkmark$$

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \qquad y = [0 \quad -1 \quad 1]$$

in diagonal canonical form

$$H_2(s) = C(sI - A)^{-1} B$$

$$= [0 \quad -1 \quad 1] \begin{bmatrix} \frac{1}{s+2} & 0 & 0 \\ 0 & \frac{1}{s+0.5} & 0 \\ 0 & 0 & \frac{1}{s-0.5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= [0 \quad -1 \quad 1] \begin{bmatrix} \frac{1}{s+2} \\ \frac{1}{s+0.5} \\ \frac{1}{s-0.5} \end{bmatrix} = -\frac{1}{s+0.5} + \frac{1}{s-0.5}$$

$$= \frac{-s+0.5+s+0.5}{s^2 - 0.25} = \frac{1}{s^2 - 0.}$$

$H_1(s) = H_2(s)$ Then these system representations are zero-state equivalent //

(b) These systems are not equivalent since at least one of the eigenvalues are different

for system 1

$\lambda_1 = -1, \lambda_2 = 0.5, \lambda_3 = -0.5$

for system 2

$\lambda_1 = -2, \lambda_2 = 0.5, \lambda_3 = -0.5$

different

hence they are not equivalent systems

(Q3) $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} a \\ t \end{bmatrix} u$

$$M = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau)\, d\tau$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \phi(t, \tau) = \begin{bmatrix} e^{(t-\tau)} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\phi(t_0, \tau) = \begin{bmatrix} e^{(t_0-\tau)} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\phi(t_0, \tau) B(\tau) = \begin{bmatrix} e^{(t_0-\tau)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ \tau \end{bmatrix} = \begin{bmatrix} a e^{(t_0-\tau)} \\ \tau \end{bmatrix}$$

$$M = \int_{t_0}^{t_1} \begin{bmatrix} a e^{(t_0-\tau)} \\ \tau \end{bmatrix} \begin{bmatrix} a e^{(t_0-\tau)} & \tau \end{bmatrix}$$

$$M = \int_{t_0}^{t_1} \begin{bmatrix} a^2 e^{2(t_0-\tau)} & a\tau e^{(t_0-\tau)} \\ a\tau e^{(t_0-\tau)} & \tau^2 \end{bmatrix} d\tau$$

$$M = \begin{bmatrix} a^2 e^{2t_0} \int_{t_0}^{t_1} e^{-\tau} d\tau & a e^{t_0} \int_{t_0}^{t_1} \tau e^{-\tau} d\tau \\ a e^{t_0} \int_{t_0}^{t_1} \tau e^{-\tau} d\tau & \int_{t_0}^{t_1} \tau^2 d\tau \end{bmatrix}$$

$$\int \tau e^{-\tau} d\tau$$

$$= -(\tau e^{-\tau} + e^{-\tau})$$

$$\frac{d}{dt}\left[-(\tau e^{-\tau} + e^{-\tau})\right]$$

$$-\left[e^{-\tau} - \tau e^{-\tau} - e^{-\tau}\right]$$

$$= \tau e^{-\tau}$$

$$M = \begin{bmatrix} a^2 e^{2t_0}\left(-e^{-\tau}\Big|_{t_0}^{t_1}\right) & -a e^{t_0}(\tau e^{-\tau}+\tau)\Big|_{t_0}^{t_1} \\ \\ -a e^{t_0}\left(\tau e^{-\tau}+\tau\right)\Big|_{t_0}^{t_1} & \frac{\tau^3}{3}\Big|_{t_0}^{t_1} \end{bmatrix}$$

$$M = \begin{bmatrix} a^2 e^{2t_0}\left(e^{-t_0} - e^{-t_1}\right) & (-a e^{t_0})\left[t_1 e^{-t_1} + t_1 - t_0 e^{-t_0} - t_0\right] \\ \\ -a e^{t_0}\left[t_1 e^{-t_1} + t_1 - t_0 e^{-t_0} - t_0\right] & \frac{1}{3}(t_1^3 - t_0^3) \end{bmatrix}$$

$$\det(M) = \frac{a^2}{3} e^{2t_0}\left(e^{-t_0} - e^{-t_1}\right)(t_1^3 - t_0^3) - a^2 e^{2t_0}\left[t_1(e^{-t_1}+1) - t_0(e^{-t_0}+1)\right]^2$$

$$\det(M) = a^2 e^{2t_0}\left(\frac{1}{3}(t_1^3 - t_0^3)(e^{-t_0} - e^{-t_1}) - \left(t_1(e^{-t_1}+1) - t_0(e^{-t_0}+1)\right)^2\right)$$

if $t_0 = 0$

$$\det(M) = a^2\left(\frac{1}{3}(t_1^3)(1 - e^{-t_1}) - \left(t_1(e^{-t_1}+1)\right)^2\right)$$

if $a \neq 0$ completely controllable