Observable Canonical Form

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots + a_{n-1} s + a_n}$$

$$Y(s)\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots + a_{n-1} s + a_n\right) = U(s)\left(b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n\right)$$

Divide every part by s^n

$$Y(s)\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots + \frac{a_n}{s^n}\right) = \left(\frac{b_1}{s} + \frac{b_2}{s^2} + \dots + \frac{b_n}{s^n}\right) U(s)$$

$$Y(s) = \left(\frac{b_1}{s} + \frac{b_2}{s^2} + \dots + \frac{b_n}{s^n}\right) U(s) - \left(\frac{a_1}{s} + \frac{a_2}{s^2} + \dots + \frac{a_n}{s^n}\right) Y(s)$$

$$Y(s) = \frac{1}{s}\left\{ b_1 U(s) - a_1 Y(s) + \frac{1}{s}\left\{ b_2 U(s) - a_2 Y(s) + \frac{1}{s}\left\{ \dots + \frac{1}{s}\left\{ b_n U(s) - a_n Y(s) \right\} \dots \right\}\right\}\right\}$$

Let's define the states as follows:

$$X_1(s) = \frac{1}{s}\left\{ b_n U(s) - a_n Y(s) \right\}$$

$$X_2(s) = \frac{1}{s}\left\{ b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s) \right\}$$

$$X_i(s) = \frac{1}{s}\left\{ b_{n-i+1} U(s) - a_{n-i+1} Y(s) + X_{i-1}(s) \right\}$$

$$X_n(s) = \frac{1}{s}\left\{ b_1 U(s) - a_1 Y(s) + X_{n-1}(s) \right\}$$

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and

$$X_n(s) = Y(s) \xrightarrow{\mathcal{L}^{-1}} x_n(t) = y(t)$$

so output of state space representation can be written as

$$y(t) = (0\ 0\ 0 \ldots \ldots 0\ 1)\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$X_1(s) = \frac{1}{s}\{b_n U(s) - a_n Y(s)\}$$

$$\downarrow$$

$$sX_1(s) = b_n U(s) - a_n \underbrace{Y(s)}_{X_n(s)} \xrightarrow{\mathcal{L}^{-1}} \dot{x}_1(t) = b_n u(t) - a_n x_n(t)$$

$$X_2(s) = \frac{1}{s}\{b_{n-1} U(s) - a_{n-1} \underbrace{Y(s)}_{X_n(s)} + X_1(s))$$

$$sX_2(s) = \{b_{n-1} U(s) - a_{n-1} X_n(s) + X_1(s)\} \xrightarrow{\mathcal{L}^{-1}} \dot{x}_2 = b_{n-1} u(t) - a_{n-1} x_n(t) + x_1(t)$$

$$X_i(s) = \frac{1}{s}\{b_{n-1+i} U(s) - a_{n-1+i} Y(s) + X_{i-1}(s)\}$$

$$sX_i(s) = \{b_{n-1+i} U(s) - a_{n-1+i} Y(s) + X_{i-1}(s)\} \xrightarrow{\mathcal{L}^{-1}} \dot{x}_i(t) = b_{n-1+i} u(t) - a_{n-1+i} y(t) + x_{i-1}(t)$$

$\downarrow$ or

$$\boxed{\dot{x}_i(t) = b_{n-1+i} u(t) - a_{n-1+i} x_n(t) + x_{i-1}(t)}$$

$$X_n(s) = \frac{1}{s}\left\{ b_1 U(s) - a_1 \underbrace{Y(s)}_{X_n(s)} + X_{n-1}(s) \right\}$$

$$sX_n(s) = b_1 U(s) - a_1 X_n(s) + X_{n-1}(s) \}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\dot{x}_n = b_1 u(t) - a_1 x_n(t) + x_{n-1}(t)$$

Now let's combine the state-equations in a matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & & 0 & -a_n \\ 1 & 0 & \cdots & & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ & & \ddots & 0 & & \vdots \\ & 0 & & 1 & 0 & -a_2 \\ & & & & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix} u(t)$$

and $y(t) = (0\ 0\ \ldots\ 0\ 1) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Block Diagram of canonical forms

(1) diagonal canonical form

$$T(s)=\frac{Y(s)}{U(s)}=\frac{b_1 s^{n-1}+b_2 s^{n-2}+\dots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\dots+a_{n-1}s+a_n}$$

Let's obtain the states

$$X_1(s)=\frac{\alpha_1}{s-\lambda_1}U(s) \qquad sX_1(s)-\lambda_1X_1(s)=\alpha_1U(s)$$

$$\boxed{X_1(s)=\frac{1}{s}\left[\lambda_1X_1(s)+\alpha_1U(s)\right]}$$

$\alpha_i \to$ residue

$\lambda_i \to$ eigenvalue

$$X_i(s)=\frac{\alpha_i}{s-\lambda_i}U(s)$$

$Y(s)=X_1(s)+X_2(s)+\dots+X_n(s)$ $\qquad sX_i(s)-\lambda_iX_i(s)=\alpha_iU(s) \qquad i=1\dots,n$

$$\boxed{X_i(s)=\frac{1}{s}\left[\lambda_iX_i(s)+\alpha_iU(s)\right]}$$

Block diagram