Ex(1):

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{B} u \qquad y = [1\ 0\ 0\ 1]\, x$$

A is in jordan form

(a) Is this system representation controllable

Using Hautus test

$$\left[\lambda I - A \,\vdots\, B\right] = \left[\begin{array}{cccc:c} \lambda-1 & -1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -1 & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

Find the eigenvalues

$$sI - A = \begin{bmatrix} s-1 & -1 & 0 & 0 \\ 0 & s-1 & -1 & 0 \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & s-2 \end{bmatrix} \Rightarrow \det(sI-A) = (s-1)^3(s-2)$$

eigenvalues $s_1 = 1 \quad s_2 = 2$
$m_1 = 3$

for $s_1 = \lambda_1 = 1$

$$\left[\lambda_1 I - A \,\vdots\, B\right] = \begin{array}{c} \left[\begin{array}{cccc:c} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \\ \begin{array}{ccccc} & \uparrow & \uparrow & \uparrow & \uparrow \\ & c_1 & c_2 & c_3 & c_4 \end{array} \end{array}$$

Number of linearly independent columns is equal to 4

(c_1, c_2, c_3, c_4) are linearly independent (rank of matrix 4)

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Hence for $\lambda = \lambda_1 = 1$ the system is controllable

For $s_2 = \lambda_2 = 2$

$$[\lambda_2 I - A \,\vdots\, B] = \begin{array}{c} \left[\begin{array}{cccc:c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & & \uparrow \\ c_1 & c_2 & c_3 & & c_4 \end{array} \end{array}$$

Number of linearly independent columns is equal to 4

(c_1, c_2, c_3, c_4) are linearly independent (rank of matrix is 4)

Hence for $\lambda = \lambda_2 = 2$ the system is controllable

* Since for both eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ the system is controllable $\Longrightarrow$ this system is completely controllable

(b) Is this system representation observable

Using Hautus test for observability

$$\begin{bmatrix} \lambda I - A \\ \hdashline C \end{bmatrix} = \left[\begin{array}{cccc} \lambda-1 & -1 & 0 & 0 \\ 0 & \lambda-1 & -1 & 0 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-2 \\ \hdashline 1 & 0 & 0 & 1 \end{array}\right]$$

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For $s_1 = \lambda_1 = 1$

$$\begin{bmatrix} \lambda_1 I - A \\ \hdashline C \end{bmatrix} = \begin{array}{c} \left[\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \\ \\ \leftarrow r_3 \\ \leftarrow r_4 \end{array}$$

Number of linearly independant ~~[illegible]~~ rows is equal to 4 (r_1, r_2, r_3, r_4) are linearly independed (rank of matrix is 4)

Hence for $\lambda = \lambda_1 = 1$ the system is observable.

For $s_2 = \lambda_2 = 2$

$$\begin{bmatrix} \lambda_2 I - A \\ \hdashline C \end{bmatrix} = \begin{array}{c} \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \\ \leftarrow r_3 \\ \\ \leftarrow r_4 \end{array}$$

Number of linearly independant rows is equal to 4 (r_1, r_2, r_3, r_4) are linearly independant (rank of matrix is 4)

Hence for $\lambda = \lambda_2 = 2$ the system is observable

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Ex(2):

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} a \\ b \\ c \\ k \end{bmatrix} u \qquad y = \begin{bmatrix} d & e & f & m \end{bmatrix} x$$

(a) What condition is necessary such that system representation is completely controllable?

Use Hautus test

characteristic equation $= d(s) = (s-1)^3 (s-2)$

$\lambda_1 = 1 \quad \lambda_2 = 2$ (eigenvalues)

For $\lambda_1 = 1$

$$[\lambda_1 I - A \vdots B] = \begin{bmatrix} 0 & -1 & 0 & 0 & a \\ 0 & 0 & -1 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & -1 & k \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ c_1 & c_2 & c_3 & c_4 \end{array}$$

c_1, c_2, c_3, c_4 are columns which are non-zero vectors

As seen (c_1, c_2, c_3) are linearly independent to have (c_1, c_2, c_3, c_4) to be linearly independent, in $\begin{bmatrix} a \\ b \\ c \\ k \end{bmatrix}$ vector $c \neq 0$. So if $c \neq 0$, whatever the value of a, b and k, the columns c_1, c_2, c_3, c_4 will be linearly independent

Hence the matrix will be rank=4 and the modes corresponding to eigenvalue $\lambda_1=1$ will all be controllable.

For $\lambda_2=2$

$$\left[\lambda_2 I - A \vdots B\right] = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & a \\ 0 & 1 & -1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \\ \begin{array}{ccccc} c_1 & c_2 & c_3 & & c_4 \end{array} \end{array}$$

c_1, c_2, c_3, c_4 are columns which we non-zero vectors

As seen (c_1, c_2, c_3) are linear independent to have (c_1, c_2, c_3, c_4) to be linearly independent in $c_4 = \begin{bmatrix} a \\ b \\ c \\ k \end{bmatrix}$ $k \neq 0$. So if $k \neq 0$, whatever the the value of a, b, c the columns c_1, c_2, c_3, c_4 will be linearly independent.

Hence the matrix will be rank=4 and the modes corresponding to eigenvalue $\lambda_2=2$ will all be controllable.