Week 14

MECE 548
Canonical structure theorem
(Kalman)

$R = [A, B, C, D]$ (time invariant

$H(t) = C e^{At} B + D\delta(t)$ (impulse response)

Let $D = 0$ (for simplicity)

* Since D involves only a direct transmission from the input to the output it can always be inserted at the end of all the computations

* It is possible to express the state of R as

direct sum $\tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_3 \oplus \tilde{x}_4$ where $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; and $\tilde{x}_4$ are the states of four subsystem representations whose relations with the input u and output y are shown in

- Any state of the form $\tilde{x}_1 + \tilde{x}_2$ is controllable
- Any state of the form $\check{x}_2 + \check{x}_4$ is unobservable
- Any state of the form $\tilde{x}_1 + \tilde{x}_3$ is observable

Hence $\tilde{x}_1 \rightarrow$ controllable + observable $= \Sigma_{co}$

$\tilde{x}_2 \rightarrow$ controllable + unobservable $= \Sigma_{cu}$

$\tilde{x}_3 \rightarrow$ uncontrollable + observable $= \Sigma_{\not{c}o}$

$\tilde{x}_4 \rightarrow$ uncontrollable + unobservable $= \Sigma_{\not{c}u}$

c → controllable
u → unobservable
$\not{c}$ → uncontrollable
o → observable

Theorem: (Canonical structure theorem: Kalman Decomposition)

$R = [A, B, C, 0]$ be time invariant

(i) R is algebraically equivalent to

$\tilde{R} = (\tilde{A}, \tilde{B}, \tilde{C}, 0)$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = [\tilde{C}_1 \quad 0 \quad \tilde{C}_3 \quad 0]$$

(ii) The subsystem representation

$R_1 = [\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, 0] \Rightarrow \dot{x}_1 = \tilde{A}_{11} x_1 + \tilde{B}_1 u$

$y_1 = \tilde{C}_1 x_1$ is completely controllable and completely observable

(iii) $R_{12} = \left[\begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, [\tilde{C}_1 \; 0], 0 \right]$

$\Downarrow$

$\dot{x}_1 = \tilde{A}_{11} x_1 + \tilde{B}_1 u$

$\dot{x}_2 = \tilde{A}_{21} x_1 + \tilde{A}_{22} x_2 + \tilde{B}_2 u$

$y_{12} = \tilde{C}_1 x_1$ is completely controllable

(iv) The subsystem representation

$$R_{24} = \left[\begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{24} \\ 0 & \tilde{A}_{44} \end{bmatrix}, \begin{bmatrix} \tilde{B}_2 \\ 0 \end{bmatrix}, [0\ 0], 0 \right]$$

$$\dot{x}_2 = \tilde{A}_{22} x_2 + \tilde{A}_{24} x_4 + \tilde{B}_2 u$$

$$\dot{x}_4 = \tilde{A}_{44} x_4$$

$$y_{24} = 0$$ is unobservable

(v) The subsystem representation

$R_4 = [\tilde{A}_{44}, 0, 0, 0]$ is uncontrollable and unobservable

Method

(a) Set $\Sigma_{cu} \triangleq R(Q) \cap N(O)$

$R(Q)$ → range space of controllability matrix

$N(O)$ → null space of observability matrix

Σ_{cu} → controllable + unobservable

Q → controllability matrix
O → observability matrix

Σ_{cu} is a uniquely defined invariant subspace under A

(b) Pick Σ_{co} such that

$$R(Q) = \underbrace{\Sigma_{co}}_{\text{controllable + observable}} \oplus \Sigma_{cu} \longrightarrow \text{controllable + unobservable}$$

*Note: Σ_{co} is not uniquely defined

(c) Pick $\Sigma_{\bar{c}u}$
$\downarrow$
uncontrollable
+
unobservable

$$N(O) = \Sigma_{cu} \oplus \Sigma_{\bar{c}u}$$

* Note $\Sigma_{\bar{c}u}$ is uniquely defined

(d) Pick $\Sigma_{\bar{c}u}$ such that

$N(O)$

$$\Sigma_{co} + \Sigma_{cu} + \Sigma_{\bar{c}o} \oplus \Sigma_{\bar{c}u} = \Sigma$$

$R(Q)$

Note that $\Sigma_{\bar{c}o}$ is not uniquely defined

* As a basis for Σ, a union of basis of Σ_{co}, Σ_{cu}, $\Sigma_{\bar{c}o}$, $\Sigma_{\bar{c}u}$ is chosen. Let T^{-1} be the matrix whose columns consists of arbitrary selected basis vectors of Σ_{co}, Σ_{cu}, $\Sigma_{\bar{c}o}$ and $\Sigma_{\bar{c}u}$ in that order. Thus if $\tilde{x}$ denotes new representative

$x = T^{-1}\tilde{x}$

controllable states

uncontrollable states

Consider $\tilde{A}$ = Since $\Sigma = R(Q) \oplus (\Sigma_{\bar{c}o} \oplus \Sigma_{\bar{c}u})$

and since $R(Q)$ is invariant under A

$\tilde{A}_{31} = 0$, $\tilde{A}_{32} = 0$ $\tilde{A}_{41} = 0$ $\tilde{A}_{42} = 0$

unobservable states

Since $\Sigma = N(O) \oplus (\Sigma_{co} \oplus \Sigma_{\bar{c}o})$ and since $N(O)$ is invariant under A

observable states

$\check{A}_{12} = 0 \quad \check{A}_{14} = 0 \quad \tilde{A}_{32} = 0 \quad \tilde{A}_{34} = 0$

Consider $\check{B}$: Since $\Sigma = R(Q) \oplus (\underbrace{\Sigma_{\bar{c}o} + \Sigma_{\bar{c}u}}_{\text{uncontrollable states}})$

controllable states

and since $R(B) \subset R(Q)$

$Q = \begin{bmatrix} B & AB & A^2B \dots A^{n-1}B \end{bmatrix}$ (hence $R(B) \subset R(Q)$)

the images under B of all the new basis vectors of Σ line in $R(Q)$; hence $\check{B}_3 = 0$, $\check{B}_4 = 0$

Consider $\check{C}$: $\Sigma = N(O) \oplus (\Sigma_{co} \oplus \Sigma_{\bar{c}o})$

unobservable states

observable states

Since $O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ any basis vector in $N(O)$ is mapped by C into the zero vector hence

$\check{C}_2 = 0$, $\tilde{C}_4 = 0$

The stated properties of the system representatations R_1, R_{12}, R_{24}, R_4 follow immediatelly from the method of selection of the subspaces and form of their representates

$$H(t) = Ce^{At}B = \tilde{C}_1 e^{\tilde{A}_{11}t} \tilde{B}_1 = \tilde{H}_1(t)$$

impulse response

and

$$* \quad R_{13} = \left[\begin{bmatrix} \bar{A}_{11} & \bar{A}_{13} \\ 0 & \bar{A}_{33} \end{bmatrix}, \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, [\bar{C}_1 \ \bar{C}_3], D \right]$$ represents a

dynamical system equivalent to dynamical system represented by $R = (A, B, C, D)$

- R_1 is zero-state equivalent to R
- R_1 is also minimal (completely observable and completely controllable)
- R_1 and R have the same impulse response and transfer function $\hat{H}(s)$

$$H(s) = \hat{H}(s)$$

- There is no representation of order less the R_1 which has $\hat{H}(s)$ as its transfer function
- R_{13} and R have the same set of input-output pairs (they are indistinguishable)
- The set of input-output pairs specifies the subsystem Σ_{co} (controllable and observable) and $\Sigma_{\bar{c}o}$ (uncontrollable but observable). But transfer function specifies only Σ_{co}.