$w_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $w_2 = \begin{pmatrix}\frac12\\-\frac12\\1\end{pmatrix}$ $w_4 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\langle w_1, v_4\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle w_2, v_4\rangle}{\langle w_1, w_2\rangle} w_2$

$\langle w_1, w_1\rangle = 2$ $\quad \langle w_1, v_4\rangle = [1\,1\,0]\begin{bmatrix}0\\1\\0\end{bmatrix} = 1$ $\quad = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{-\frac12}{\frac32}\begin{pmatrix}\frac12\\-\frac12\\1\end{pmatrix} =$

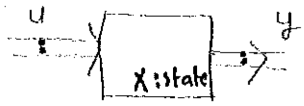
$\langle w_2, w_2\rangle = \frac{3}{2}$ $\quad \langle w_2, v_4\rangle = [\frac12\ -\frac12\ 1]\begin{bmatrix}0\\1\\0\end{bmatrix} = -\frac12$

8.00 $w_4 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac12\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{-1/2}{1.5}\begin{pmatrix}0.5\\-0.5\\1\end{pmatrix} = \begin{pmatrix}1/6\\ \cdot\\ \cdot\end{pmatrix}$ l.d. not a problem.

9.00 $\longrightarrow \{v_1, v_2, v_4\}$ l.ind. set

$= \begin{bmatrix}0-\frac12+\frac16\\ 1-\frac12-\frac16\\ 0-0+\frac13\end{bmatrix} = \begin{bmatrix}-\frac13\\ \frac13\\ \frac13\end{bmatrix}$

10.00



11.00 $\dot{X}(t) = f(X(t), t, u(t))$

Do we have a solution?
Is that solution unique?

$y(t) = g(X(t), t, u(t))$

12.00

$p(x,\cdot)$, $p(\cdot, t)$

13.00 $\dot{X}(t) = p(X, t)$, $X(t_0) = X_0$

* $\phi(t)$ is a solution if it satisfies the initial condition and the dif. equations.

14.00 $\phi(t, t_0, X_0) = \phi(t)$

$\phi(t_0, t_0, X_0) = X_0$ $\longrightarrow t$ için

15.00 $P(\cdot, \cdot): R^n \times R^+ \to R^n$ $\longrightarrow X$ için

Assumption 1.) on $p(\cdot,\cdot)$: Let X be a fixed vector then we assume that $p(x, \cdot)$ is piecewise continuous.

16.00

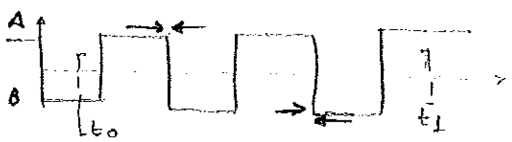
<u>Definition:</u> A function $f: R^+ \to R^n$ is said to be <u>piecewise continuous</u> iff for any finite interval $[t_0, t_1]$ $\exists$ finite number of points say $t_1', t_2', \ldots, t_k'$ such that $f(\cdot)$ is continuous for all t except possibly at $t_1', t_2', \ldots, t_k'$. Furtermore the right and left limits exists for each t_i' $i = 1, \ldots, k$.

17.00

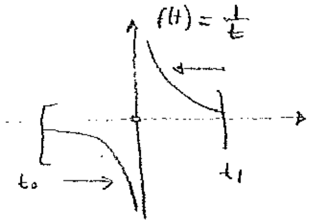
18.00

19.00

<u>Example:</u> Square wave: There are some finite countable number of discontinuities so piecewise continuous right and left limit exist for each discontinuous point



2)



Not piecewise cont. because there is no right and left limits. [if limit exits it should be a number "not infinity"]

3) $f(x) = \begin{cases} 1 & \text{if } x = \frac{a}{b} \text{ where } a, b \text{ are integers} \\ 0 & \text{otherwise} \end{cases}$ not piecewise continuous.

$\rightarrow$ there are infinite number of discontinuities

<u>Assumption 2:</u> Let t be a fixed number. Then it is assumed that $p(\cdot, t) : R^n \rightarrow R^n$ satisfies Lipschitz condition. i.e $\exists\, k(t) \geq 0$ such that $\| p(x_1, t) - p(x_2, t) \| \leq k(t) \| x_1 - x_2 \| \quad \forall t \in R^+ \quad \forall x_1, x_2 \in R^n$

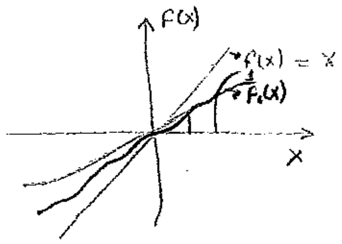
<u>Note:</u> $k(\cdot) : R^+ \rightarrow R^+$

<u>Example 1:</u> $f(x) = x$

$f : R \rightarrow R$ satisfies Lipschitz condition.

$| f(x_1) - f(x_2) | \leq k \cdot | x_1 - x_2 |$ $\qquad k(t) = k = 2$ (for example)

$f(x) = x \qquad |x_1 - x_2| \leq 2 |x_1 - x_2|$



$\dot{x} = p(x, t) \qquad x(t_0) = x_0$

(*) $\dot{x} = p(x, t) \qquad x(t_0) = x_0 \qquad$ Time interval : $[t_0, t_1]$

$x_0(t_0) = x_0$

$\dot{x}(t) = p(x, t)$

$\int_{t_0}^{t} d\, x(t') = \int_{t_0}^{t} p(x, t')\, dt'$

$x(t') \Big|_{t_0}^{t} = \int_{t_0}^{t} p(x(t'), t')\, dt'$

$x(t) - x(t_0) = \int_{t_0}^{t} p(x(t'), t')\, dt'$

$x(t) = x_0 + \int_{t_0}^{t} p(x(t'), t')\, dt' \qquad (**)$

8.00

$$X(t) \approx X_1(t) = X_0 + \int_{t_0}^{t} p(X_0, t')\,dt'$$

9.00

$$X(t) \approx X_2(t) = X_0 + \int_{t_0}^{t} p(X_1(t'), t')\,dt'$$

$$\vdots$$

10.00

$$X_k(t) = X_0 + \int_{t_0}^{t} p(X_{k-1}(t'), t')\,dt'$$

11.00

Claim 1:) $\lim_{k\to\infty} X_k(t)$ exists and is a solution of $(*)$ existance

12.00

2:) $\lim_{k\to\infty} X_k(t)$ in the only solution of $(*)$ uniqueness

13.00

Proof of Claim 1:

$$\| X_{k+1}(t) - X_k(t) \| = \| X_0 + \int_{t_0}^{t} p(X_k(t'), t)\,dt' - \cancel{X_0} - \int_{t_0}^{t} p(X_{k-1}(t'), t')\,dt' \|$$

14.00

$$= \| \int_{t_0}^{t} \left(p(X_k(t'), t) - p(X_{k-1}(t'), t')\right) dt' \|$$

15.00

$$\le \int_{t_0}^{t} \| p(X_k(t'), t') - p(X_{k-1}(t'), t') \|\,dt'$$

16.00

$$\left| \int_{t_0}^{t_1} f(t')\,dt' \right| \le \int_{t_0}^{t_1} |f(t')|\,dt'$$

17.00

$$\le \int_{t_0}^{t} k(t')\, \| X_k(t') - X_{k-1}(t') \|\,dt' \leftarrow \text{from Lipschitz condition}$$

18.00

non-negative and piecewise continuous

A piecewise cont. function is bounded on a finite interval

19.00

so $k(t') \le \bar{k} \quad \forall t' \in [t_0, t] \subset [t_0, t_1]$

$$\therefore \| X_{k+1}(t) - X_k(t) \| \le \bar{k} \int_{t_0}^{t} \| X_k(t') - X_{k-1}(t') \|\,dt'$$

Babalar Günü

MAYIS 1999

Pazartesi		3	10	17	24	31
Salı		4	11	18	25	
Çarşamba		5	12	19	26	
Perşembe		6	13	20	27	
Cuma		7	14	21	28	
Cumartesi	1	8	15	22	29	
Pazar	2	9	16	23	30	

HAZİRAN 1999

Pazartesi		7	14	21	28
Salı	1	8	15	22	29
Çarşamba	2	9	16	23	
Perşembe	3	10	17	24	
Cuma	4	11	18	25	
Cumartesi	5	12	19	26	
Pazar	6	13	20	27	

TEMMUZ 1999

Pazartesi		5	12	19	26
Salı		6	13	20	27
Çarşamba		7	14	21	28
Perşembe	1	8	15	22	29
Cuma	2	9	16	23	30
Cumartesi	3	10	17	24	
Pazar	4	11	18	25	

$$\| X_1(t) - \underbrace{X_0(t)}_{X_0} \| \leq \cancel{\bar{K} \int_{t_0}^{t} \| X_1(t') - X_0(t')} \| dt' \leq M \qquad X_1(t) = X_0 + \int_{t_0}^{t} P(X_0, t')\,dt'$$

$$\| X_1 - X_0 \| = \left\| \int_{t_0}^{t_1} P(X_0, t')\,dt' \right\| \leq \int_{t_0}^{t_1} \underbrace{\| P(X_0, t') \|\,dt'}_{M}$$

$$\| X_1 - X_0 \| \leq M$$

$$\| X_2(t) - X_1(t) \| \leq \bar{K} \int_{t_0}^{t_1} \underbrace{\| X_1(t') - X_0 \|}_{\leq M}\,dt' \leq \bar{K} \int_{t_0}^{t} M\,dt' = \bar{K} M (t - t_0)$$

$$\| X_3(t) - X_2(t) \| \leq \bar{K} \int_{t_0}^{t} \underbrace{\| X_2(t') - X_1(t') \|}_{\leq \bar{K} M (t' - t_0)}\,dt' \leq \bar{K}^2 \int_{t_0}^{t} M (t' - t_0)\,dt' = M \bar{K}^2 \left. \frac{(t' - t_0)^2}{2} \right|_{t_0}^{t}$$

$$= \frac{M \bar{K}^2}{2} (t - t_0)^2$$

$$\| X_4(t) - X_3(t) \| \leq M \bar{K}^3 \frac{(t - t_0)^3}{3!}$$

$$\vdots$$

$$\| X_{m+1}(t) - X_m(t) \| \leq M \bar{K}^m \frac{(t - t_0)^m}{m!} \qquad \text{let } T = t - t_0$$

$$\| X_{k+m} - X_k \| = \| (\underbrace{X_{k+m} - X_{k+m-1}}) + (\underbrace{X_{k+m-1} - X_{k+m-2}}) \cdots - X_k) \|$$

from triangle inequality

$$\leq \| X_{k+m} - X_{k+m-1} \| + \| X_{k+m-1} - X_{k+m-2} \| + \cdots + \| X_{k+1} - X_k \|$$

$$\frac{M (\bar{K} T)^{k+m-1}}{(k+m-1)!} + \frac{M (\bar{K} T)^{k+m-2}}{(k+m-2)!} + \cdots + \frac{M (\bar{K} T)^{k}}{k!}$$

$$= M \sum_{l=k}^{k+m-1} \frac{(\bar{K} T)^l}{l!} \leq M \sum_{l=0}^{\infty} \frac{(\bar{K} T)^l}{l!} = M e^{\bar{K} T}$$

$$\mathring{x} = p(x,t) \quad x(t_0) = x_0 \iff x(t) = x_0 + \int_{t_0}^{t} p(x(t'),t')\,dt' \quad t \in [t_0, t_1]$$

$$x_1(t) = x_0 + \int_{t_0}^{t} p(x_0, t')\,dt'$$

$$x_2(t) = x_0 + \int_{t_0}^{t} p(x_1(t'), t')\,dt'$$

$$\vdots$$

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} p(x_k(t'), t')\,dt'$$

We must prove that

(1) $\lim_{k\to\infty} x_k(t) = \phi(t)$ (2) exists, it is a continuous functions and it satisfies (3) $\phi(t) = x_0 + \int_{t_0}^{t} p(\phi(t), t')\,dt'$ and is (4) the only solution of this d.e (integral equation)

$$\| x_{k+1}(t) - x_k(t) \|$$

$$\| x_1(t) - x_0 \| = \| \int_{t_0}^{t} p(x_0, t')\,dt' \|$$

$p(x_0, t')$ is a piecewise continuous function, integrated on a finite interval $\therefore$ integral is bounded i.e $\exists M$ st

$$\| x_2(t) - x_1(t) \| = \| x_0 + \int_{t_0}^{t} p(x_1(t), t')\,dt' - x_0 - \int_{t_0}^{t} p(x_0, t')\,dt \|$$

$$= \| \int_{t_0}^{t} [\, p(x_1(t'), t') - p(x_0, t')]\,dt' \|$$

$$\le \int_{t_0}^{t} \| p(x_1(t'), t') - p(x_0, t')] \|\,dt'$$

from liphstic $\le \int_{t_0}^{t} \underbrace{k(t')}_{\text{piecewise cont.}} \| x_1(t') - x_0 \|\,dt' \le \bar{k} M (t - t_0)$

$k(t') \le \bar{k} \quad \forall (t') \in [t_0, t_1]$

$$\| x_2(t) - x_1(t) \| \le \bar{k} M (t - t_0)$$

$$\| x_3(t) - x_2(t) \| = \| \int_{t_0}^{t} (p(x_2(t'), t') - p(x_1(t'), t')\,dt' \|$$

$$\le \int_{t_0}^{t} \| p(x_2(t'), t') - p(x_1(t'), t') \|\,dt'$$

$$\le \underbrace{k(t')}_{\bar{k}} \underbrace{\| x_2(t') - x_1(t') \|}_{\bar{k} M (t - t_0)}$$

$$M (\bar{k})^2 \int_{t_0}^{t} (t' - t_0)\,dt' = M (\bar{k})^2 \frac{(t - t_0)^2}{2}$$

$$\| x_{n+1}(t) - x_n(t) \| \le M \cdot (\bar{k})^n \frac{(t - t_0)^n}{n!}$$

The aim was to prove that $x_n(t)$ is a convergent sequence.
First we will prove that it is cauchy.

$$\|X_{n+m}(t) - X_n(t)\| = \|X_{n+m}(t) - X_{n+m-1}(t) + X_{n+m-1}(t) - X_{n+m-2}(t) + X_{n+m-2}(t) - X_{n+m-3}(t) \cdots X_{n+1}(t) - X_n(t)\|$$

$$\|X_{n+m}(t) - X_n(t)\| \le \|X_{n+m}(t) - X_{n+m-1}(t)\| + \dots + \|X_{n+1}(t) - X_n(t)\|$$

$$\le M\cdot(\bar{k})^{n+m-1}\frac{(t-t_0)^{n+m-1}}{(n+m-1)!} + \dots + M(\bar{k})^n\frac{(t-t_0)^n}{n!}$$

$0 \le t-t_0 \le t_1-t_0 = T$

$$\|X_{n+m}(t) - X_n(t)\| \le M\sum_{l=0}^{m-1}\frac{(\bar{k})^{l+n}\,T^{l+n}}{(l+n)!} = M(\bar{k}T)^n\sum_{l=0}^{m-1}\frac{(\bar{k}T)^l}{(l+n)!}$$

Note that $(n+l)! > (n!).(l!)$

$$\frac{1}{(n+l)!} < \frac{1}{n!}\cdot\frac{1}{l!}$$

$$\therefore \|X_{n+m}(t) - X_n(t)\| \le \frac{M(\bar{k}T)^n}{n!}\underbrace{\sum_{l=0}^{m-1}\frac{(\bar{k}T)^l}{l!}}_{< e^{\bar{k}T}}$$

$$\boxed{\|X_{n+m}(t) - X_n(t)\| \le M e^{\bar{k}T}\frac{(\bar{k}T)^n}{n!}}$$

Note: $\lim_{n\to\infty}\frac{(\bar{k}T)^n}{n!} = 0$ This implies that $\lim_{n\to\infty}\|X_{n+m}(t) - X_n(t)\| \to 0$

$\therefore \{X_n(t)\}$ is a Cauchy sequence.

Note: The vector space of continuous functions with $\|\cdot\|_\infty$ defined on it is a <u>complete</u> space. Note also that $X_n(t)$ is a c.funct since it is integral of a piece wise continuous func. $p(x(t'),t')$ so aplying the theorem above, $\lim_{n\to\infty} X_n(t) = \phi(t)$ and $\phi(t)$ is continuous.
continuous

$$X_{n+1}(t) = X_0 + \int_{t_0}^{t} P(X_n(t'),t')\,dt' \qquad X_n(t) \to \phi(t)$$
$\Downarrow \; \phi(t)$

$$\| X_0 + \int_{t_0}^{t} P(X_n(t'),t')\,dt' - (X_0 + \int_{t_0}^{t} P(\phi(t'),t')\,dt')\| < \varepsilon$$

$$\| \int_{t_0}^{t} [P(X_n(t'),t') - P(\phi(t'),t')]\,dt' \|$$

$$\leq \int_{t_0}^{t} \| P(X_n(t'),t') - P(\phi(t'),t')\|\,dt'$$

$$< \int_{t_0}^{t} k(t') \cdot \underbrace{\| X_n(t') - \phi(t')\|}_{<\varepsilon_0}\,dt' \leq \bar{k} \cdot \varepsilon_0\, T = \varepsilon$$

for a given $\varepsilon > 0$
$\therefore$ choose n large enough $\ni$ the corresponding to satisfies
$\bar{k}\, T\, \varepsilon_0 < \varepsilon$

$\therefore$ right hand side converges to $X_0 + \int_{t_0}^{t} P(\phi(t'),t')\,dt'$
left " " " $\phi(t)$

$$\phi(t) = X_0 + \int_{t_0}^{t} P(\phi(t'),t')\,dt'$$

$\phi(t) = \lim_{n\to\infty} X_n(t)$ is a solution of de $\dot{X}(t) = P(X,t)$ $X(t_0) = X_0$

· Uniqueness of the solution:

<u>Bellman Gronwall Lemma:</u>

Let $c_1 \geq 0$ be a non negative real number
Let $u(\cdot)$ and $k(\cdot)$ be real valued piecewise continuous functions on R_+. Let $k(\cdot) \geq 0$ on R_+. If

$$u(t) \leq c_1 + \int_{t_0}^{t} k(t')u(t')\,dt' \quad \forall t \in R^+$$

then $u(t) \leq c_1 \cdot e^{\int_{t_0}^{t} k(t')\,dt'}$

$\dot{U}(t) = k(t)\,u(t)$ ← küçük
↳ büyük

<u>Proof:</u> $U(t) \triangleq c_1 + \int_{t_0}^{t} k(t')u(t')\,dt'$ $\qquad u(t) \leq U(t)$

$$\left(k(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'}\right) u(t) \leq \left(k(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'}\right) U(t)$$

$$\underline{\dot{U}(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'} - U(t)\,k(t)\,e^{-\int_{t_0}^{t} k(t')\,dt'}} \leq 0$$

$$\int_{t_0} \frac{d}{dt}\left[U(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'}\right] \leq \int_{t_0} 0$$

$$\frac{d}{dt}\left[U(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'}\right]$$

$$\int_{t_0} \frac{d}{dt}\left[U(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'}\right] dt \; \int_{t_0}^{t} d\,d$$

$$\mathcal{U}(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'} - \underbrace{\mathcal{U}(t_0)}_{c_1} \underbrace{e^{-\int_{t_0}^{t_0} k(t')\,dt'}}_{1} \leq 0$$

$$\mathcal{U}(t)\, e^{-\int_{t_0}^{t} k(t')\,dt'} \leq c_1$$

$$\mathcal{U}(t) \leq c_1 \cdot e^{\int_{t_0}^{t} k(t')\,dt'}$$

but $u(t) \leq \mathcal{U}(t) \Rightarrow \boxed{u(t) \leq c_1\, e^{\int_{t_0}^{t} k(t')dt'}}$

Uniqueness

Proof of uniqueness is by contradiction. Suppose that we have two solutions $\phi(t)$, $\psi(t)$. Then

$$\phi(t) = x_0 + \int_{t_0}^{t} p(\phi(t'), t')\,dt'$$
$$\psi(t) = x_0 + \int_{t_0}^{t} p(\psi(t'), t')\,dt'$$

two solutions

$$\|\phi(t) - \psi(t)\| = \left\| \int_{t_0}^{t} [p(\phi(t'), t') - p(\psi(t'), t')]\,dt' \right\|$$

$$\leq \int_{t_0}^{t} \|p(\phi(t'), t') - p(\psi(t'), t')\|\,dt'$$

$$\underbrace{\|\phi(t) - \psi(t)\|}_{\text{continuous}} \leq \int_{t_0}^{t_1} \underbrace{k(t')}_{\substack{\text{non-negative}\\ \text{piecewise cnt}}} \underbrace{\|\phi(t') - \psi(t')\|}_{\text{continuous}}\,dt' + \underbrace{c_1}_{\substack{\text{arbitrary}\\ \text{non negative}}} \quad \text{from Liph.}$$

$$u(t) \leq c_1 + \int_{t_0}^{t_1} k(t')\, u(t')\,dt' \quad \text{from Bellman Gronwall}$$

$$\|\phi(t) - \psi(t)\| \leq c_1 \cdot e^{\int_{t_0}^{t} k(t')\,dt'} \quad \forall c_1 \geq 0$$

let $c_1 = 0 \Rightarrow 0 \leq \|\phi(t) - \psi(t)\| \leq 0$

$$\|\phi(t) - \psi(t)\| = 0 \Rightarrow \phi(t) - \psi(t) = 0 \qquad \phi(t) = \psi(t)$$

8.00

Obtain x_n by Picard's iteration

↓

9.00

Find a bound on $\|x_{n+1} - x_n\|$

↓

10.00

Find a bound on $\| x_{n+m} - x_n \|$

↓

11.00

Show that x_{n+m} is a cauchy

↓

12.00

Argue that $x_n(t)$ is continuous and $\|\cdot\|_\infty, F^n[t_0,t_1]$ is complete $\Rightarrow$ x_n is convergent $\lim x_n \to 0$

↓

Show that ϕ satisfies $\phi(t) = x_0 + \int_{t_0}^{t} f(\phi(t'), t')\,dt'$

↓

Prove that $\phi(t)$ is unique — B.G lemma

Dynamical Systems:

13.00

Example

14.00

$V_R = R \cdot i_R$

$C \cdot \frac{dV_c}{dt} = i_c$

4 PAZAR

15.00

$$C\,\frac{dV_c}{dt} = \frac{V_R}{R} = \frac{u(t) - V_c(t)}{R}$$

16.00

$$C\,\frac{dy}{dt} = \frac{1}{R}\,u(t) - \frac{1}{R}\,y(t)$$

17.00

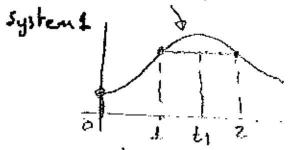
$$\begin{cases} \frac{dy}{dt} = \frac{-1}{RC}\,y(t) + \frac{1}{RC}\,u(t) \\ y(t_0) = V_c(t_0) \end{cases}$$

$y(t) = u(t+t_1)$

18.00

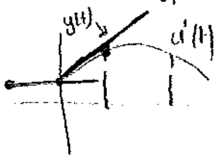
not causal not dynamical system

System 1

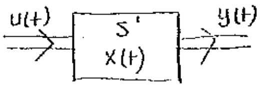


19.00

$y(t) = \frac{d}{dt} u(t)$



$\Rightarrow$ it has a dynamical system

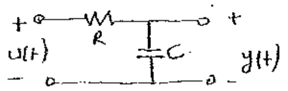


T : time $T = \mathbb{Z}$ · Set of integers
$= \mathbb{R}$: Set of real numbers

$u \in U$ = Set of all admissable inputs. $\{u(t)$: Value of the input at time $t\} = \mathcal{U}$
usually $\mathcal{U} = R^m$

$y \in \mathcal{Y}$: Set of all possible output function $\{y(t)\} = Y \rightarrow$ an example $\mathbb{R}^r$

Σ : State (usually R^n). $x(t) \in \Sigma$



$u(\cdot) \rightarrow [t_0, t]$

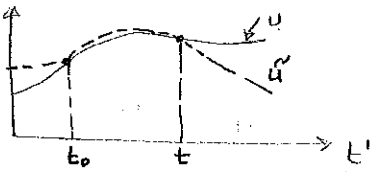
State transition function: $S(t, t_0, x_0, u(\cdot)) = x(t)$

final time, initial time, initial state, input, final state

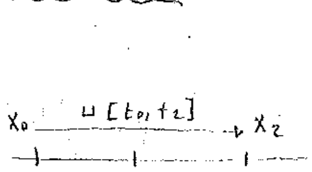
Read-out map: $y(t) = r(t, x(t), u(t))$

<u>State transition axiom:</u> Given two inputs $u(\cdot)$, $\tilde{u}(\cdot)$. Suppose that $u(t') = \tilde{u}(t')$ $\forall\, t' \in [t_0, t]$. Then a representation is said to satisfy the state transition axiom if

$$S(t, t_0, x_0, u(\cdot)) = \tilde{S}(t, t_0, x_0, \tilde{u}(\cdot))$$



<u>Semi Group Axiom:</u> Let $t_0 \leq t_1 \leq t_2$ $S(t_2, t_0, x_0, u(\cdot)_{[t_0,t_2]}) =$
$S(t_2, t_1, s(t_1, t_0, x_0, u(\cdot)_{[t_0,t_1]}), u(\cdot)_{[t_1,t_2]})$



$$s(t_1, t_0, x_0, u(\cdot)_{[t_0,t_1]}) = x_1$$

A Representation of the Form:
$D = (T, U, \mathcal{Y}, \Sigma, s, r)$ where
T: time , U: Set of all possible inputs $\mathcal{Y}$: Set of all possible outputs

Σ : Set of states

$s : T \times T \times \Sigma \times \mathcal{U} \rightarrow \Sigma$ ie $x(t) = s(t, t_0, x_0, u(\cdot))$ is the state transition function

$r : t \times \Sigma \times U \rightarrow Y$ $\quad y(t) = r(t, x(t), u(t))$ is the read-out map

Is called a dynamical system representation if 's' satisfies the transition axiom and semigroup axiom

Example:

$$\xrightarrow{u} \boxed{S} \xrightarrow{y} \qquad y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} u(\tau)\, d\tau$$

we want to build a dynamical system

Solution: $D = (T, \mathcal{U}, \mathcal{Y}, \Sigma, s, r)$

$T = R$ (Real numbers) continuous $\quad$ (Z) discrete

$\mathcal{U}$: Set of all continuous functions

$u : T \rightarrow R$

$U = R \quad , \quad \Sigma = R$

$\mathcal{Y}$: Set of all continuous functions mapping $\underset{T}{\overset{}{R}} \rightarrow \underset{Y}{R}$

$Y = R$

$x(t) = s(t, t_0, x_0, u_{[t_0,t]}(\cdot))$ State is not unique.

$$x_0 \triangleq x(t_0) \triangleq \int_{-\infty}^{t_0} e^{-(t_0-\tau)} u(\tau)\, d\tau \quad \text{(it is our choise)}$$

$$x(t) = \int_{-\infty}^{t} e^{-(t-\tau)} u(\tau)\, d\tau = \underbrace{\int_{-\infty}^{t_0} e^{-(t-\tau)} u(\tau)\, d\tau}_{\neq x_0} + \int_{t_0}^{t} e^{-(t-\tau)} u(\tau)\, d\tau$$

$$A : \int_{-\infty}^{t_0} e^{-t_0} e^{t_0} e^{-t} e^{\tau} u(\tau)\, d\tau = e^{t_0 - t} \underbrace{\int_{-\infty}^{t_0} e^{-(t_0-\tau)} u(\tau)\, d\tau}_{x_0}$$

$$x(t) = e^{-(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{-(t-\tau)} u(\tau)\, d\tau$$

$x(t) = s(t, t_0, x_0, u[t_0,t]) \Rightarrow$ state transition axiom is satisfied

Semi group axiom

$$t_0 \longrightarrow t_1 (x_1) \longrightarrow t_2 (x_2)$$

$$X(t_1) = S(t_1, t_0, X_0, u)$$
$$= e^{-(t_1-t_0)} X_0 + \int_{t_0}^{t_1} e^{-(t_1-z)} u(z)\, dz$$

$$X(t_2) = e^{-(t_2-t_1)} X_1 + \int_{t_1}^{t_2} e^{-(t_2-z)} u(z)\, dz$$

$$= e^{-(t_2-t_1)} \left[e^{-(t_1-t_0)} X_0 + \int_{t_0}^{t_1} e^{-(t_1-z)} u(z) dz \right] + \int_{t_1}^{t_2} e^{-(t_2-z)} u(z)\, dz$$

$$X(t_2) = e^{-(t_2-t_0)} X_0 + \int_{t_0}^{t_1} e^{-(t_2-z)} u(z)\, dz + \int_{t_1}^{t_2} e^{-(t_2-z)} u(z)\, dz$$

$$= e^{-(t_2-t_0)} X_0 + \int_{t_0}^{t_2} e^{-(t_2-z)} u(z)\, dz = S(t_2, t_0, X_0, u(\cdot)_{[t_0,t_2]})$$

$$\frac{y(t) = X(t)}{T,\ U,\ Y,\ \Sigma,\ S,\ r}$$

Both of the axioms are checked (This is a dynamical system)

Dynamical System

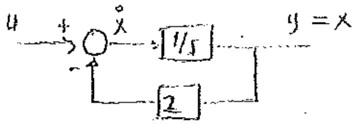
$\exists\ [T, U, \Sigma, Y,\ s(t, t_0, x_0, u(\cdot)) = x(t),\ \ r(t, x(t), u(t)) = y(t)]$

1) If $u(t') = \tilde{u}(t') \quad \forall\ t' \in [t_0, t]$

then $s(t, t_0, x_0, u(\cdot)) = s(t, t_0, x_0, \tilde{u}(\cdot))$

2) $S(t_2, t_1, \underbrace{s(t_1, t_0, x_0, u(\cdot))}_{x(t_1)}; u(\cdot)) = S(t_2, t_0, x_0, u(\cdot))$

Example:



a) $\dot{x} = u - 2x$

b) $T = R^+$ Continuous time

c) U = Set of all piecewise cont. functions that map R^+ into R. $u \in U \Rightarrow$
$u: \underset{T}{R^+} \to \underset{U}{R}$, u is piecewise cont

$\Sigma = R$

$$x(t) = \int_{t_0}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau = e^{-2t} \int_{t_0}^{t} e^{2\tau} u(\tau)\, d\tau$$

$$\Rightarrow \frac{d}{dt}(x(t)) = -2e^{-2t} \int_{t_0}^{t} e^{2\tau} u(\tau)\, d\tau + (e^{-2t})\, e^{2\tau} u(\tau) \Big|_{\tau=t}$$

8.00

d) Y = Set of all continuous functions $y : R^+ \to R$, $Y = R$

e) $s = ?$

9.00

$$y(t) = r(t, x(t), u(t)) \qquad x(t) = e^{-2(t-t_0)} x_0 + \int_{t_0}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau$$

10.00

$$x(t_0) = e^{-2 \cdot 0} x_0 + \underbrace{\int_{t_0}^{t_0} e^{-2(t-\tau)} u(\tau)\, d\tau}_{0} = x_0$$

11.00

$$\frac{d\,x(t)}{dt} = -2e^{-2(t-t_0)} x_0 + \left[e^{-2(t-\tau)} u(\tau) \Big|_{\tau=t} + \int_{t_0}^{t} -2\, e^{-2(t-\tau)} u(\tau)\, d\tau \right.$$

12.00

$$= -2 \Big[\underbrace{e^{-2(t-t_0)} + \int_{t_0}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau}_{x(t)} \Big] + u(t) \qquad \dot{x}(t) = -2x(t) + u(t)$$

13.00

$$x(t) = s(t, t_0, x_0, u) = e^{-2(t-t_0)} x_0 + \int_{t_0}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau$$

14.00

$$y(t) = r(t, x(t), u(t)) = x(t)$$

15.00

Axioms: let $u(t') = \tilde{u}(t') \quad \forall\, t' \in [t_0, t]$

16.00

1) $$s(t, t_0, x_0, u(\cdot)) = e^{-2(t-t_0)} x_0 + \int_{t_0}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau$$
$$s(t, t_0, x_0, \tilde{u}(\cdot)) = e^{-2(t-t_0)} x_0 + \int_{t_0}^{t} e^{-2(t-\tau)} \tilde{u}(\tau)\, d\tau$$
equal to each other since $u = \tilde{u}$ when $\tau \in [t_0, t]$!

17.00

2) $$x(t_1) = e^{-2(t_1-t_0)} x_0 + \int_{t_0}^{t_1} e^{-2(t_1-\tau)} u(\tau)\, d\tau$$

18.00

$$\text{LHS}: \quad s(t_2, t_1, x_1, u) = e^{-2(t_2-t_1)} x(t_1) + \int_{t_1}^{t_2} e^{-2(t_2-\tau)} u(\tau)\, d\tau$$

19.00

$$\text{LHS} = e^{-2(t_2-t_1)} \left[e^{-2(t_1-t_0)} x_0 + \int_{t_0}^{t_1} e^{-2(t_1-\tau)} u(\tau)\, d\tau \right] + \int_{t_1}^{t_2} e^{-2(t_2-\tau)} u(\tau)\, d\tau$$
$$= e^{-2(t_2-t_0)} x_0 + \int_{t_0}^{t_1} \underbrace{e^{-2(t_2-t_1+t_1-\tau)} u(\tau)}_{e^{-2(t_2-\tau)} u(\tau)}\, d\tau + \int_{t_1}^{t_2} e^{-2(t_2-\tau)} u(\tau)\, d\tau$$

HAZİRAN 1999

Pazartesi		7	14	21	28
Salı	1	8	15	22	29
Çarşamba	2	9	16	23	
Perşembe	3	10	17	24	
Cuma	4	11	18	25	
Cumartesi	5	12	19	26	
Pazar	6	13	20	27	

TEMMUZ 1999

Pazartesi		5	12	19	26
Salı		6	13	20	27
Çarşamba		7	14	21	28
Perşembe	1	8	15	22	29
Cuma	2	9	16	23	30
Cumartesi	3	10	17	24	
Pazar	4	11	18	25	

AĞUSTOS 1999

Pazartesi		2	9	16	23	30
Salı		3	10	17	24	
Çarşamba		4	11	18	25	
Perşembe		5	12	19	26	
Cuma		6	13	20	27	
Cumartesi		7	14	21	28	
Pazar	1	8	15	22	29	

$y(t) = \int u(t-x)$

$$= e^{-2(t_2-t_0)} X_0 + \int_{t_0}^{t_2} e^{-2(t_2-\tau)} u(\tau)\, d\tau = RHS = s(t_2, t_0, X_0, u)$$

Both axioms are satisfied ∴ this is a dynamical system representation.

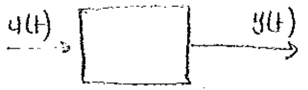
$t = 4 \quad \lfloor t \rfloor = 3$

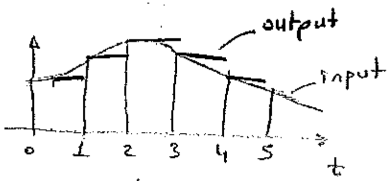
örnek

$t = 4.8 \quad \lfloor t \rfloor = 4$

$t = 5.1 \quad \lfloor t \rfloor = 5$

Example:





$y(t) = u(\lfloor t \rfloor) \qquad t \in [\lfloor t \rfloor, \lfloor t \rfloor + 1] \qquad \lfloor t \rfloor$: greatest integer smaller than t

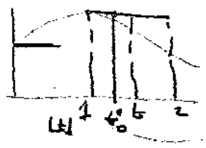
$T = R^+$

$\mathcal{U}$: Set of all continuous function that map R^+ into R $\quad u \in \mathcal{U} \Rightarrow u: R^+ \to R$

$U = R \qquad u$ is continuous

$\mathcal{Y}$: Set of all piecewise ~~constant~~ continuous function $y: R^+ \to R$, $Y = R$

$$X(t) = s(t, t_0, X_0, u(\cdot)) = \begin{cases} u(\lfloor t \rfloor) & \text{if } \lfloor t \rfloor \geq t_0 \\ X_0 & \text{if } \lfloor t \rfloor < t_0 \end{cases}$$



Axioms:

$s(t, t_0, X_0, u) = s(t, t_0, X_0, \tilde{u})$ if $u(t') = \tilde{u}(t') \quad \forall t' \in [t_0, t]$

$$\begin{cases} u(\lfloor t \rfloor) & \text{if } \lfloor t \rfloor \geq t_0 \\ X_0 & \text{otherwise} \end{cases} \stackrel{?}{=} \begin{cases} \tilde{u}(\lfloor t \rfloor) \geq t_0 \Leftrightarrow \lfloor t \rfloor \in [t_0, t] \\ X_0 \quad \text{otheswise} \end{cases}$$

$\lfloor t \rfloor \in [t_0, t] \Rightarrow u(\lfloor t \rfloor) = \tilde{u}(\lfloor t \rfloor)$ ∴ we have the same state

state transition axiom ✓

2^{nd} axiom

$s(t_2, t_1, X_1, u) = s(t_2, t_0, X_0, u) \qquad X_1 = s(t_1, t_0, X, u)$

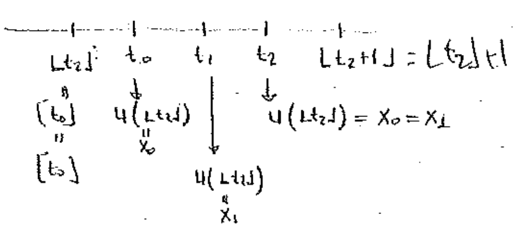
örnek

$t = 0$

$0 \leq t < 1 \quad y(t) = u(0)$

$1 \leq t < 2 \quad y(t) = u(1)$

Case 1



$\Sigma = R$

8.00 Case 2

9.00

$t_0 \quad \lfloor t_1 \rfloor \quad t_1 \quad t_2$

10.00 $u(\lfloor t_2 \rfloor) = s(t_2, t_1, x_1, u) = x_1 = u(\lfloor t_2 \rfloor)$

$x_1 = u(\lfloor t_2 \rfloor) = s(t_2, t_0, x_0, u) = u(\lfloor t_2 \rfloor)$

11.00

Case 3:

12.00

$t_0 \quad t_1 \quad \lfloor t_2 \rfloor \quad t_2$

$s(t_2, t_1, x_1, u(\cdot)) = u(\lfloor t_2 \rfloor)$

13.00 $s(t_2, t_0, x_0, u(\cdot)) = u(\lfloor t_2 \rfloor)$

14.00 $\therefore$ 2^{nd} axiom is satisfied

25 PAZAR

15.00 $\lfloor t_2 \rfloor = \lfloor 3.7 \rfloor = 3$

Case 1:

$3 \quad t_0 \quad t_1 \quad t_2 \quad 4$

$\lfloor t_1 \rfloor \quad 3.2 \quad 3.5 \quad 3.7$

$$x_2(t) \begin{cases} s(t_2, t_1, x_1, u[t_1, t_2]) = \begin{cases} u(\lfloor t_2 \rfloor) & t_1 < \lfloor t_2 \rfloor \\ x_1 & \text{otherwise} \end{cases} \\ s(t_2, t_0, x_0, u[t_0, t_2]) = \begin{cases} u(\lfloor t_2 \rfloor) & t_0 < \lfloor t_2 \rfloor \\ x_0 & \text{otherwise} \end{cases} \end{cases}$$

16.00

Note that: $\lfloor t_2 \rfloor = \lfloor t_1 \rfloor = \lfloor t_0 \rfloor = \lfloor 3.7 \rfloor = \lfloor 3.5 \rfloor = \lfloor 3.1 \rfloor$

17.00 $x_1 = u(\lfloor t_1 \rfloor) = x_0 = u(\lfloor t_0 \rfloor)$

$\therefore x_1 = x_0$ so for this case axiom 2 holds.

18.00

Case 2:

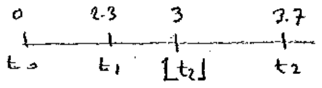
19.00 $0 \quad 3 \quad 3.5 \quad 3.7$

$t_0 \quad \lfloor t_2 \rfloor \quad t_1 \quad t_2$

$$x_2(t) = \begin{cases} s(t_2, t_1, x_1, u) = \begin{cases} u(\lfloor t_2 \rfloor) & t_1 < \lfloor t_2 \rfloor \\ x_1 & \text{else} \end{cases} = x_1 \\ \qquad = u(u\lfloor t_2 \rfloor) \\ s(t_2, t_0, x_0, u) = \begin{cases} u(\lfloor t_2 \rfloor) & \text{if } t_0 < \lfloor t_2 \rfloor \\ x_0 & \text{otherwise} \end{cases} = u(\lfloor t_2 \rfloor) \end{cases}$$

$x_1 \geq x_1$

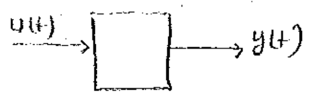
Case 3:

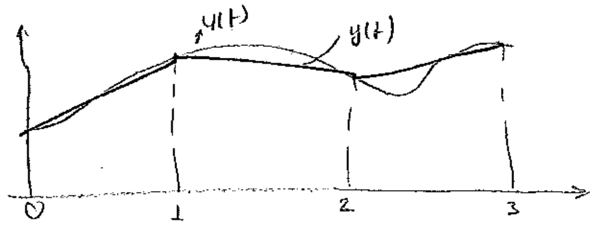


$$X_2(t) = S(t_2, t_1, X_1, u) = \begin{cases} u(\lfloor t_2 \rfloor) & \text{if } \lfloor t_2 \rfloor > t_1 \\ X_1 & \text{otherwise} \end{cases} = u(\lfloor t_2 \rfloor)$$

$$X_2(t) = S(t_2, t_0, X_0, u) = \begin{cases} u(\lfloor t_2 \rfloor) & \text{if } \lfloor t_2 \rfloor > t_0 \\ X_0 & \text{otherwise} \end{cases} = u(\lfloor t_2 \rfloor)$$

Example 4:



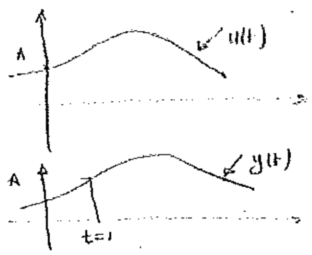


Future value effects present values.

This is not a dynamical system

bu sistemde bir önceki dereceli noktadan bir sonrakine eğim çiziliyor
0'dan 1'e
1'den 2'ye
dinamik sistem değil

Example 5:



T: R

u(t) : Set of all continuous functions $u: R \to R = U$

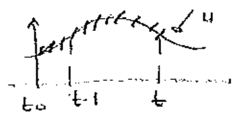
$y(t) = U$, $Y = R$

$\Sigma = R$

$y(t) = x(t) = u(t-1)$ X

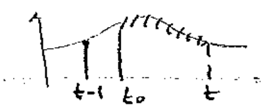
Case 1:

$x(t) = S(t, t_0, x_0, u) = u(t-1) \quad t_0 < t-1$



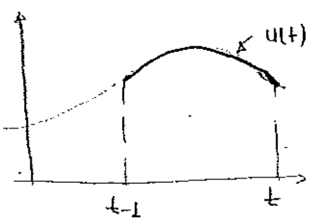
Case 2:

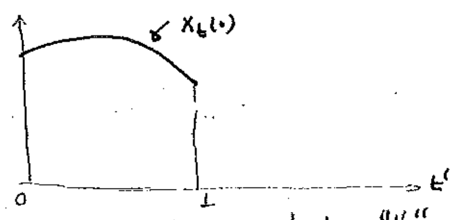
$t_0 > t-1$



$x(t) = u(t-1)$ is not possible

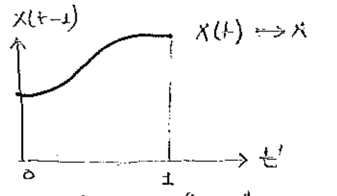
We cannot define $x(t-1)$ in this case.



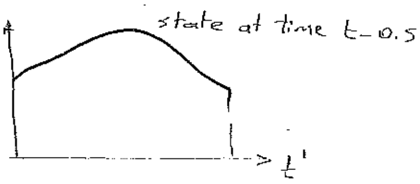


State at time "t"

$X(t) \to X_t$



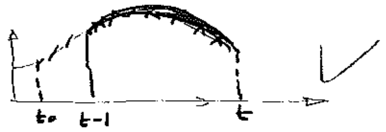
State "t-1"



$S(t, t_0, X_0, u) = ?$

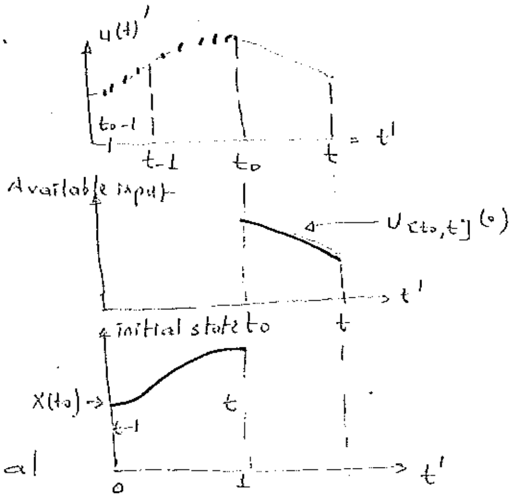
Case 1:

$t_0 < t-1$



$S(t, t_0, X_0, u) = D^{[0,1]}_{t-1}\, u$

Case 2: $t-1 < t_0$



$S(t, t_0, X_0, u) =$

1) Shift the input waveform so that $t_0 \to 1$
2) Define the waveform of concatination of the initial state and the state available input
3) Shift the resultant waveform such that $t \to 1$

Σ: Function space: Set of all continuous functions that map $[0,1] \to R$.

read out map

$y(t) = u(t-1) = X_t(t')\big|_{t'=0}$

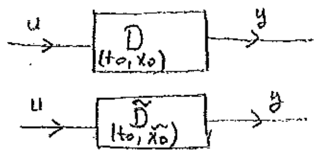
1) State transition axiom
2) Semigroup axioms.

} Prove that two axioms hold.

Equivalance: Equivalant states: Let D and $\tilde{D}$ be two dynamical systems representations. Assume that they have the same input and output spaces. i.e.: $\mathcal{U} = \tilde{\mathcal{U}}$ and $\mathcal{Y} = \tilde{\mathcal{Y}}$. The states X_0 and $\tilde{X}_0$ are said to be equivalent if the two systems produce the same output $y(t)$ and $\tilde{y}(t)$, If same input $u_{[t_0,t_1]}(\cdot)$ is applied and they started from the initial states X_0 and $\tilde{X}_0$

Equivalance of Dynamical System Representation:

Let $D=(T, U, y, \Sigma, s, r)$ and $\tilde{D}=(T, U, y, \tilde{\Sigma}, \tilde{s}, \tilde{r})$ are said to be equivalent if and only if $\forall t_0 \in T$ $\forall x_0 \in \Sigma$ there exists a $\tilde{x}_0 \in \tilde{\Sigma}$ which is equivalent to x_0 and similarly $\forall \hat{x}_0 \in \tilde{\Sigma}$ $\exists x_0 \in \Sigma$ which is " to $\tilde{x}_0$ at t_0.



Example:

System: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{B} u$

$y = \underbrace{(1 \quad 1)}_{c} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ State-space rep) (not dynamical rep

$T: R^+$

U: Set of all piecewise cont. functions $u: R^+ \to R \Rightarrow U = R$

y: " " continuous " $y: R^+ \to R \Rightarrow Y = R$

$\Sigma = R^2$

$$x(t) = s(t, t_0, x_0, u) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$\dot{x}_1(t) = x_1 + u(t) \Rightarrow x_1(t) = e^{(t-t_0)} x_{01} + \int_{t_0}^{t} e^{(t-\tau)} u(\tau)\, d\tau$$

$$\dot{x}_2(t) = \alpha x_2 + u(t) \Rightarrow x_2(t) = e^{\alpha(t-t_0)} x_{02} + \int_{t_0}^{t} e^{\alpha(t-\tau)} u(\tau)\, d\tau$$

Does it satisfies the two axioms? Yes

Note: Show that both axioms are satisfied.

$y(t) = r(t, x(t), u(t)) = x_1(t) + x_2(t) = C \cdot x(t)$ read out map.

Let $\alpha = 1$

$$y(t) = e^{(t-t_0)}(x_{01} + x_{02}) + 2\int_{t_0}^{t} e^{(t-\tau)} u(\tau)\, d\tau$$

Define a new system $\tilde{D}$ as follows

$$\tilde{D} = (T, U, y, \tilde{\Sigma}, \tilde{s}, \tilde{r})$$

$$\tilde{s}(t, t_0, \tilde{x}_0, u) = e^{(t-t_0)}\tilde{x}_0 + \int_{t_0}^{t} e^{(t-\tau)}\, 2 \cdot u(\tau)\, d\tau$$

$$\tilde{r}(t, \tilde{x}(t), u(t)) = \dot{x}_1(t) + \dot{x}_2 = \tilde{x} + 2u$$

$$\tilde{\Sigma} = R \qquad y = \tilde{x}$$

$$\boxed{\dot{\tilde{x}} = \tilde{x} + 2u \quad y = \tilde{x} \quad \dot{\tilde{x}} = \dot{x}_1 + \dot{x}_2}$$

D		$\tilde{D}$
$\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$	$\longleftrightarrow$	$(x_{01} + x_{02})$

order not same but these two systems are equivalent

Time Invariant Dynamical System:

$$y(t) = r(t, x(t), u(t)) = \underbrace{r(t, s(t, t_0, x_0, u(\cdot)), u(t))}$$

$\rho(t, t_0, x_0, u(\cdot))$: response function.

Example: In the previous example read out map was $y(t) = \tilde{x}(t)$ and the response function was $y(t) = e^{(t-t_0)}\tilde{x}_0 + \int_{t_0}^{t} e^{(t-\tau)}\, 2\, u(\tau)\, d\tau$

Definition:

A dynamical system is said to be time invariant iff we can associate with it a representation $D = (T, U, y, \Sigma, s, r)$ such that

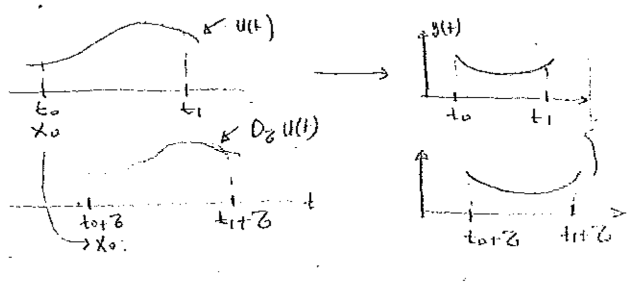
a) U is closed under translation

b) $\forall\, t_0, t_1 \in T$ with $t_1 \geq t_0$, for all $x_0 \in \Sigma$, $\forall u \in U$

$$\rho(t_1, t_0, x_0, u) = \rho_1(t_1 + \tau, t_0 + \tau, x_0, D_\tau u(\cdot))$$

where $D_\tau u$ is the shifted input by an amount τ.

translation yapıldığında aynı domain'de u(t) tanımlanabilmeli



8.00 Example: $\dot{x}(t) = x(t) + e^{2t} u(t)$ $\qquad y(t) = e^{-2t} x(t)$

$T = \mathbb{R}$

9.00 U: set of all pie. cont. function

Y: set of all cont. functions

10.00 Σ: $\mathbb{R}$

$$x(t) = e^{(t-t_0)} x_0 + \int_{t_0}^{t} e^{(t-t')} e^{2t'} u(t')\, dt'$$

11.00 Response functions.

$$y(t) = e^{-2t} e^{(t-t_0)} x_0 + \int_{t_0}^{t} e^{-2t} e^{(t-t')} e^{2t'} u(t')\, dt'$$

12.00

$$= e^{-(t+t_0)} x_0 + \int_{t_0}^{t} e^{(t-t')} \cdot e^{-2(t-t')} u(t')\, dt'$$

13.00

$$\rho(t, t_0, x_0, u(\cdot)) = e^{-(t+t_0)} x_0 + \int_{t_0}^{t} e^{-(t-t')} u(t')\, dt'$$

not equal

14.00

$$\rho(t+\tau, t_0+\tau, x_0, D_\tau u(\cdot)) \overset{?}{=} e^{-(t+\tau+t_0+\tau)} x_0 + \int_{t_0+\tau}^{t+\tau} e^{-(t+\tau-t')} \overbrace{u(t'-\tau)}^{D_\tau u} dt'$$

$\tilde{t} \triangleq t' - \tau. \quad d\tilde{t} = dt'$

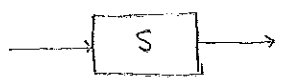
15.00

$$\int_{t_0}^{t} e^{-(t-\tilde{t})} u(\tilde{t})\, d\tilde{t}$$

same

This representation is time varying

16.00 Example:

17.00 

$u(\cdot)$: "String of letters" 'a, b, c'

18.00 A typical input: aaabaccbbcbbba...

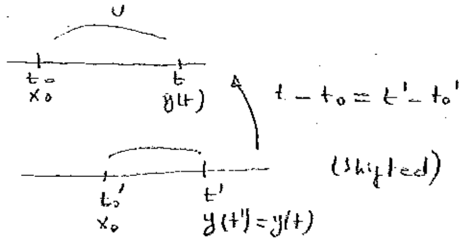
S: counts the # of 'a's in this string. the output is a non-negative integer

19.00

Find a dy. system rep. is it time var. or invar.

response funct.

$$y(t) = \rho(t, t_0, x_0, u(\cdot)) = r(t, s(t, s(t, t_0, x_0, u(\cdot)), u(t))$$

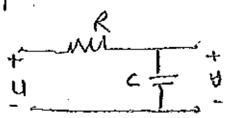


Linear System:

Definition: A system representation $D = (T, U, y, \Sigma, s, r)$ is said to be linear if
a) U, y and Σ are linear spaces over the same field F
b) $\rho(t, t_0, \alpha x_0 + \beta \bar{x}_0, \alpha u_1(\cdot) + \beta u_2(\cdot)) = \alpha \rho(t, t_0, x_0, u_1(\cdot)) + \beta \rho(t, t_0, \bar{x}_0, u_2(\cdot))$

hold. This can be written as: $\rho : \Sigma \times U \to y$ is a linear map

Examples:



$$c \frac{dV_c}{dt} = \frac{u - y}{R} = -\frac{y}{R} + \frac{u}{R}$$

$$\frac{dV_c}{dt} = -\frac{y}{RC} + \frac{u}{RC}$$

$T = \mathbb{R}$

U : Set of all piecewise continuous functions mapping $T \to U$ ($T = \mathbb{R}$, $U = \mathbb{R}$)

y : " " " continuous " " $T \to Y$ ($T = \mathbb{R}$, $Y = \mathbb{R}$)

$\Sigma : \mathbb{R}$, $y(t) = x(t)$: $r(t, x(t), u(t))$

(*) $s(t, t_0, x_0, u) = e^{-1/RC (t - t_0)} (x_0) + \int_{t_0}^{t} e^{-\frac{1}{RC}(t - \tau')} \frac{1}{RC} u(t') dt'$

Semi group axiom;

is satisfied since the output depends only on $U_{[t_0, t]}(\cdot)$

from equation (*)

State transition axiom:

$s(t_2, t_0, x_0, u) = s(t_2, t_1, s(t_1, t_0, x_0, u), u)$

<u>Is this representation Linear?</u>

Answer:

$U, y, \Sigma \leftarrow \Sigma = \mathbb{R}$ <u>Field: $\mathbb{R}$</u>

U: A subspace of function space <u>Field: $\mathbb{R}$</u>

y: A subspace of function space <u>Field: $\mathbb{R}$</u>

$\therefore$ U, y, Σ are vector spaces over the same field $\mathbb{R}$.

8.00
$$\rho(t, t_0, X_0, u) = y(t) = \underbrace{e^{-\frac{1}{RC}(t-t_0)} X_0}_{\text{zero input response}} + \underbrace{\int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} u(t')\, dt'}_{\text{zero state response}}$$

Linearlik 9.00 sorusu
$$\rho(t, t_0, \alpha X_0 + \beta \bar{X}_0, \alpha u + \beta \bar{u}) = e^{-\frac{1}{RC}(t-t_0)} (\alpha X_0 + \beta \bar{X}_0) + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} [\alpha u(t') + \beta \bar{u}(t')]$$

10.00
$$= \alpha\left[e^{-\frac{1}{RC}(t-t_0)} X_0 + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} u(t')\, dt'\right]$$
$$+ \beta\left[e^{-\frac{1}{RC}(t-t_0)} \bar{X}_0 + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} \bar{u}(t')\, dt'\right]$$

11.00

$= \alpha . A + \beta C$ is a linear.

12.00

A system is linear only input-output relationsheep is linear

13.00 and initial condition is zero. In here, we cannot say like this.

As the initial " is not zero. We must attend initial condition

14.00 $u(t) = 2 . u(t)$ $y(t) = 2 . y(t)$ $X(0) = 0$ 1. situation.

$u(t) = 2 . u(t)$ $y(t) \neq 2 . y(t)$ $X(0) \neq 0$ 2. situation.

15.00

Is this representation time-invariant.

16.00
$$\rho(t, t_0, X_0, u) = e^{-\frac{1}{RC}(t-t_0)} X_0 + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} u(t')\, dt'$$

a) U is closed under translations

17.00 If $u \in U \Rightarrow P_T u \in U$

b) $t_0 \to t_0 + T$ $\rightarrow y(t+T) = e^{-\frac{1}{RC}(t+T-t_0-T)} X_0 + \int_{t_0+T}^{t+T} e^{-\frac{1}{RC}(t+T-t')} \frac{1}{RC} u(\underbrace{t'-T}_{\tilde{t}=t'-T})$

18.00 $t \to t+T$

$u(t) \to u(t-T)$ $\rightarrow y(t) = e^{-\frac{1}{RC}(t-t_0)} X_0 + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-t')} \frac{1}{RC} u(t')\, dt'$

19.00 $X_0 \to X_0$

$$y(t+T) = e^{-\frac{1}{RC}(t-t_0)} X_0 + \int_{t_0}^{t} e^{-\frac{1}{RC}(t-\tilde{t})} \frac{1}{RC} u(\tilde{t})\, d\tilde{t} = y(t)$$

$\therefore$ time invariant

Example:

$\xrightarrow{u} \boxed{M} \xrightarrow{y}$

$\mathcal{U}$: Strings of 'a', 'b' and 'c'   i.e. a typical input is aabbbcccbcaa

$u(t) = \{ a \text{ or } b \text{ or } c \}$

$y(t)$ = number of 'a's in a given string starting from $t=-\infty$

T: $\mathbb{Z}$: set of integers

$\mathcal{U}$: Strings of letters a, b, c

U : $\{a, b, c\}$

$\mathcal{Y}$: A non-negative non-decreasing sequences of integers

Y : $\mathbb{Z}$ non-negative integers

Σ :

$y(t) = x(t) = s(t, t_0, x_0, u(\cdot)) = x_0 +$ numbers of 'a's in $u_{[t_0,t]}(\cdot)$

Typical input

$\cdots aa\overset{\downarrow t_0}{c}ccc\ bb\ a\ b\ a\ \overset{\downarrow t}{a}\ a\ bb\ c\ c$

$x_0 = 5$ $x(t) = 8$

Exercise: Prove that the two axioms hold.

Is this system linear? ⟵ (α ile çarpmanın anlamı yok)

No, Input $\mathcal{U}$ is not a vector space

Is it time invariant? Yes. (Prove: exercise)

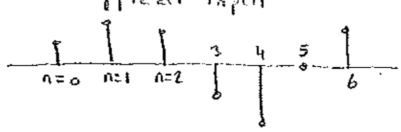
↳ durumlar incele

Example:

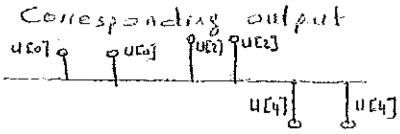
$$y[n] = \begin{cases} u[n] & n = \text{even} \\ u[n-1] & n = \text{odd} \end{cases}$$

$\xrightarrow{u[n]} \boxed{} \xrightarrow{y[n]}$

Typical input



Corresponding output



T : $\mathbb{Z}$

$\mathcal{U}$: Sequence of real numbers

U : $\mathbb{R}$

$\mathcal{Y}$: Sequence of real number

Y : $\mathbb{R}$

8.00

$X[n] \triangleq u[n-1]$

$$y[n] = r(n, X[n], u[n]) = \begin{cases} u[n] & \text{if } n \text{ is even} \\ X[n] & \text{if } n \text{ is odd} \end{cases} = \begin{cases} u[n] & \text{if } n \text{ is even} \\ u[n-1] & \text{if } n \text{ is odd} \end{cases}$$

9.00

$$S[n, n_0, x_0, u] = X[n] = u[n-1]$$

10.00

$$= \begin{cases} u[n-1] & \text{if } n-1 \geq n_0 \equiv n \geq n_0+1 \equiv n > n_0 \\ X_0 & \text{if } n-1 < n_0 \\ & \quad n < n_0+1 \\ & \quad \text{or } n = n_0 \end{cases}$$

11.00

Exercise: Proof that both axioms hold

Linearity:

12.00

a) U: sequence of real numbers, vector space field $= R$

Y: " " " " "

13.00

$\Sigma = R$

U, Y, Σ are vector spaces on the field $F = R$

14.00

$$\rho(n, n_0, X_0, u) = \begin{cases} \left.\begin{array}{l} u[n] \text{ if } n \text{ is even} \\ u[n-1] \text{ if } n \text{ is odd} \end{array}\right\} n_0 < n \\ \begin{array}{l} u[n] \text{ if } n \text{ is even} \\ X_0 \text{ if } n \text{ is odd} \end{array} \quad n_0 = n \end{cases}$$

15.00

Case 1: $n_0 < n$

16.00

$$\rho(n, n_0, \alpha X_0 + \beta \bar{X}_0, \alpha u + \beta \bar{u}) = \begin{cases} \alpha u[n] + \beta \bar{u}[n] & \text{if } n \text{ is even} \\ \alpha u[n-1] + \beta \bar{u}[n-1] & \text{if } n \text{ is odd} \end{cases}$$

17.00

$$= \alpha \rho(n, n_0, X_0, u) + \beta \rho(n, n_0, \bar{X}_0, \bar{u})$$

Case 2:

18.00

$n_0 = n$

$$\rho(n, n_0, \alpha X_0 + \beta \bar{X}_0, \alpha u + \beta \bar{u}) = \begin{cases} \alpha u[n] + \beta \bar{u}[n] & \text{if } n \text{ is even} \\ \alpha X_1 + \beta \bar{X}_0 & \text{if } n \text{ is odd} \end{cases}$$

19.00

$$\alpha \rho(n, n_0, X_0, u(\cdot)) + \beta \rho(n, n_0, \bar{X}_0, \bar{u}(\cdot))$$

$\therefore$ this system is linear

$$y(n) = \begin{cases} u[n] & n \text{ even} \\ y[n-1] & n \text{ odd} \end{cases}$$

$u[t-\tau]$, $u[y] = 1$

$y[n] \cdot r[n]$, $r[n] = n$

$r[n-1] = n-1$

Exercise: Show that it is time varying. One simple example is enough to prove that it is not time invariant system.

$A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ Representation:

$$\left.\begin{aligned} \dot{X}(t) &= A(t)\,X(t) + B(t)\,u(t) \\ y(t) &= C(t)\,X(t) + D(t)\,u(t) \end{aligned}\right\} \text{ a dynamical representation}$$

$A(t)$: is an $n \times n$ real Matrix

$B(t)$: is an $n \times r$ " "

$C(t)$: is an $m \times n$ " "

$D(t)$: is an $m \times r$ " "

Piecewise continuous functions of time.

Example:

$\dot{X} = -\frac{1}{RC} X + \frac{1}{RC} u(t)$ $\qquad A(t) = -\frac{1}{RC}$ $(n=1)$

$y = X$ $\qquad B(t) = \frac{1}{RC}$ $(r=1)$

$C(t) = 1$ $(m=1)$

$D(t) = 0$

$\dot{X}(t) = A(t)\,X(t) + B(t)\,u(t)$

$\dot{X}(t) = A(t)\,X(t)$: Homogeneous differential equation

$\dot{X}(t) = \underbrace{A(t)\,X(t)}_{P(X,t)}$ From Existance and uniqueness theorem

1.) $P(X, \cdot)$: must be piecewise continuous function.

$A(t) \cdot X$: piecewise continuous function of 't'.

2.) $\| P(X_1,t) - P(X_2,t) \| \le k(t) \cdot \| X_1 - X_2 \|$ Lipschitz.

$\| A(t)\,X_1 - A(t)\,X_2 \| = \| A(t)(X_1 - X_2) \| \le \underbrace{\| A(t) \|}_{k(t)} \cdot \| X_1 - X_2 \|$

$k(t) \ge 0$, $k(t)$ is piecewise continuous

$\Rightarrow$ $\dot{X}(t) = A(t)\,X(t)$ $\quad X(t_0) = X_0$ has a unique solution $\phi(t, t_0, X_0)$

Fact 1: $\phi(t, t_0, X_0)$ is a linear function of X_0. i.e. $\phi(t, t_0, \alpha X_0 + \beta \bar{X}_0)$

$= \underbrace{\alpha\, \phi(t, t_0, X_0) + \beta\, \phi(t, t_0, \bar{X}_0)}_{RHS}$

Proof:

We will show that RHS satisfies the equation

$$\frac{d}{dt}\left[\phi(t, t_0, \alpha X_0 + \beta \bar{X}_0) \right] = \alpha \frac{d}{dt}\, \phi(t, t_0, X_0) + \beta \frac{d}{dt}\, \phi(t, t_0, \bar{X}_0)$$

$$= \alpha \cdot A(t)\,\phi(t,t_0,X_0) + \beta A(t)\,\phi(t,t_0,\bar{X}_0)$$

$$= A(t)\left[\alpha\,\phi(t,t_0,X_0) + \beta\,\phi(t,t_0,\bar{X}_0)\right]$$

RHS satisfies the initial condition

$$\alpha\,\underbrace{\phi(t_0,t_0,X_0)}_{X_0} + \beta\,\underbrace{\phi(t_0,t_0,\bar{X}_0)}_{\bar{X}_0} = \alpha X_0 + \beta \bar{X}_0$$

uniqueness of the solution $\Rightarrow$ $RHS = \phi(t,t_0,\alpha X_0 + \beta\bar{X}_0)$

$$\therefore\ \phi(t,t_0,X_0) = \Phi(t,t_0)\cdot X_0 = X(t)$$

Example:

$\dot{X} = X(X-1)$ $\quad t_0 = 0 \quad X(0) = 0.5 \;\Big|\; t_0 = 0 \quad \bar{X}(0) = 2$ $\quad t$

$\dfrac{dx}{X(X-1)} = dt \qquad \left(\dfrac{-1}{X} + \dfrac{1}{X-1}\right)dx = dt \qquad -\ln x + \ln(x-1)\Big|_{t_0} = t - t_0$

$$X(t) = \frac{c}{c - e^t} \qquad t = 0 \quad \frac{c}{c-1} = 0.5 \qquad \frac{c}{c-1} = 2$$

$$X(t) = \frac{1}{e^t + 1}\ , \quad \bar{X}(t) = \frac{1}{1 - e^t}$$

$\ln \dfrac{x-1}{x}\Big|_{t_0} = t - t_0$

$$\boxed{X(t) = \frac{X_0}{1 - (X_0 - 1)e^t}} \qquad \text{non-linear}$$

$\ln\left(\dfrac{\frac{X(t)-1}{X(t)}}{\frac{X_0-1}{X_0}}\right) = t - t_0$

Example

$\ln\left\{\dfrac{X(t)-1}{X(t)}\cdot\dfrac{X_0}{X_0-1}\right\} =$

$$\dot{X} = \alpha X \qquad X(t) = e^{\alpha t} X_0 \qquad \text{linear}$$

---

(1) $\dot{X}(t) = A(t)\cdot X(t)\ , \quad X(t_0) = X_0 \qquad X(t) \in R^n$ (vector)

Solution: $\phi(t,t_0,X_0)$: linear function of X_0

$$-\ln(x) + \ln(x-1)\Big|_{X_0}^{X} = t - t_0 \qquad \left[-\ln(x) + \ln(x-1)\right] - \left[-\ln(x_0) + \ln(x_0 - 1)\right]$$

$$\ln\frac{x-1}{x} - \frac{\ln(2-1)}{\ln(2)} = t - 0$$

$$\ln\frac{x-1}{x} = t$$

$$\frac{x-1}{x} = e^t$$

$$x - 1 = xe^t$$

$$x[1 - e^t] = 1$$

② $\dot{X}(t) = A(t)\ X(t)$ $\quad X(t)$: $n\times n$ real matrix

Definition: A solution $X(t)$ of ② is called a fundamental solution if $\det(X(t_0)) \neq 0$

Property: Let $X(t)$ be a fundamental matrix then $\det X(t) \neq 0$ $\forall t$

Proof: Proof is by contradiction.

Assume that $\det(X(t_1)) = 0$. This means that columns of $X(t_1)$ are linearly dependent. $\exists\, c \in R^n$ s.t $c \neq \theta_n$

$$X(t_1)\cdot c = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

Claim:

Consider the vector differential equation ① start from the initial condition $(\underbrace{X(t_1)c}_{\theta_n}, t_1)$ then $x(t) = X(t)\cdot c$ is the solution i.e

$\frac{d}{dt} X(t)\cdot c = A(t)\cdot X(t)\cdot c = \dot{x}(t)$ and $X(t_0)\cdot c = \underbrace{x(t_0)}_{x(t_0)} \neq \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$

We have two solutions of eq ①

1) $x_1(t) = X(t)\cdot c \neq \theta_n$ $\forall t$ in particular $X(t_0)\cdot c = x_0 \neq \theta_n$

2) $x_2(t) = \theta_n$ $\forall t$.

Note that: $x_1(t_1) = x_2(t) = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$ and $x_1(t_0) \neq x_2(t_0)$ $\Big\}$ contradicting the uniqueness of solution of differential equation

Problem: Suppose that $X(t)$ is given. Find the solution of

$$\dot{x}(t) = A(t)\cdot x(t), \quad x(t_0) = x_0$$

① denklem

$x_0 = \alpha_1 X_1(t_0) + \alpha_2 X_2(t_0) + \dots + \alpha_n X_n(t_0)$ $\quad X(t) = [\,x_1(t) \dots x_n(t)\,]$

$x_0 = [\,x_1(t_0) \dots x_n(t_0)\,]\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} = X(t_0)\cdot\alpha \Rightarrow \alpha = X^{-1}(t_0)\cdot x_0$

$x(t) = X(t)\cdot\alpha$: solution. Since ① $\frac{d}{dt}x(t) = \frac{d}{dt}X(t)\,\alpha = A(t)\underbrace{X(t)\cdot\alpha}_{x(t)}$

$= A(t)\cdot x(t)$

② $x(t_0) = X(t_0)\cdot\alpha = X(t_0)\cdot X^{-1}(t_0)\cdot x_0 = x_0$ ✓

it is difficult to calculate this. Because of this we will define a new matrix.

$\boxed{\text{Definition:}}$ A matrix $\Phi(t,t_0)$ is said to be the state-transition matrix if

1) $\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ – it is unique

2) $\Phi(t_0,t_0) = I$

Note that 1) $\det\Phi(t,t_0) \neq 0$ $\quad\forall t,\ \forall t_0 \in R$

Note 2: the solution of

$$\dot{X}(t) = A(t)\,X(t) \qquad X(t_0) = X_0 \quad \text{is} \quad X(t) = \Phi(t,t_0)\,X_0$$

Example: $\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} X(t)$ our aim is to find the state transition matrix

$$\dot{X}(t) = A \cdot X(t) \qquad X(0) = I \Rightarrow \Phi(t,0)$$

$$sX(s) - \underbrace{I}_{\text{initial state}} = A\,X(s)$$

$$(sI - A)\,X(s) = I$$

$$X(s) = (sI - A)^{-1}$$

$$\mathcal{L}^{-1}(sI - A)^{-1} = \Phi(t,0)$$

$$(sI-A)^{-1} = \begin{bmatrix} s-1 & 0 \\ -2 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{2}{(s-1)(s+1)} & \frac{1}{(s+1)} \end{bmatrix}$$

$$\mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{2}{(s-1)(s+1)} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ e^{t}-e^{-t} & e^{-t} \end{bmatrix} \qquad \Phi(t,0) = \begin{bmatrix} e^{t} & 0 \\ e^{t}-e^{-t} & e^{-t} \end{bmatrix}$$

$$\frac{1}{s-1} - \frac{1}{s+1}$$

Calculation of $\Phi(t,t_0)$

$$\Phi(t,t_0) = \Phi(t,0)\,\Phi(0,t_0) = \Phi(t,0)\cdot\Phi^{-1}(t_0,0)$$

Fact 1: $\Phi(t,t_0) = \underbrace{\Phi(t,t_1)\,\Phi(t_1,t_0)}_{RHS} \quad \forall\, t, t_0, t_1$

Proof:

$$\frac{d}{dt}(RHS) = \frac{d}{dt}\left[\Phi(t,t_1)\,\Phi(t_1,t_0)\right]$$

$$= A(t)\,\underbrace{\Phi(t,t_1)\,\Phi(t_1,t_0)}$$

$$= A(t)\cdot(RHS) \qquad \text{let } t = t_1$$

$$\Phi(t_1,t_0) \overset{?}{=} RHS\Big|_{t=t_1} = \underbrace{\Phi(t_1,t_1)}_{I}\,\Phi(t_1,t_0)$$

$\Phi(t_1,t_0) \overset{\checkmark}{=} \Phi(t_1,t_0)$ $\therefore$ Because of uniqueness of solutions of dif. equation $RHS = \Phi(t,t_0)$

Fact 2: $\Phi(t_0,t_1) = \Phi^{-1}(t_1,t_0)$

Proof: By fact 1:

$$\Phi(t,t_0)\Big|_{t=t_0} = \Phi(t,t_1)\,\Phi(t_1,t_0)\Big|_{t=t_0}$$

$$I = \Phi(t_0,t_1)\cdot\Phi(t_1,t_0) \Rightarrow \text{result}$$

8.00

For example

$$\Phi(t,t_0) = \Phi(t,0)\cdot\Phi(0,t_0) = \begin{bmatrix} e^{t-t_0} & 0 \\ e^{t-t_0} - e^{-(t-t_0)} & e^{-(t-t_0)} \end{bmatrix}$$

9.00

$$\dot{X}(t) = A(t)\cdot X(t)$$

$X(t)$: Fundemental matrix

10.00

$\Phi(t,t_0)$: State transition matrix

$$\begin{cases} \frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0) \\ \Phi(t_0,t_0) = I \end{cases}$$

11.00

Example:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

12.00

$$\dot{x}_2(t) = t\cdot x_2(t)$$

13.00

$$\int_{x_2(t_0)}^{x_2(t)} \frac{dx_2}{x_2} = \int_{t_1}^{t} t'\cdot dt' \implies \ln x_2(t) - \ln x_2(t_0) = \frac{t^2}{2} - \frac{t_0^2}{2}$$

14.00

$$\frac{x_2(t)}{x_2(t_0)} = e^{t^2/2 - t_0^2/2}$$

15.00

$$x_2(t) = e^{t^2/2 - t_0^2/2}\cdot x_2(t_0)$$

$$\dot{x}_1(t) = x_2(t)$$

16.00

$$x_1(t) - x_1(t_0) = \int_{t_0}^{t} e^{\tau^2/2 - t_0^2/2}\, x_2(t_0)\, d\tau$$

17.00

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} e^{\tau^2/2 - t_0^2/2}\, x_2(t_0)\, d\tau$$

18.00

$$\Phi(t,t_0) = \begin{bmatrix} 1 & \int_{t_0}^{t} e^{\tau^2/2 - t_0^2/2}\, d\tau \\ 0 & e^{t^2/2 - t_0^2/2} \end{bmatrix}$$

19.00

$$\phi(t,t_0) = \begin{bmatrix} 1 & \int_{t_0}^{t} e^{\tau^2/2 - t_0^2/2} \\ 0 & e^{t^2/2 - t_0^2/2} \end{bmatrix}\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} \cdot \phi(t,t_0)$$

Zafer Bayramı

AĞUSTOS 1999

Pazartesi		2	9	16	23	30
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EYLÜL 1999

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EKİM 1999

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Fact: If A is a constant matrix, then $\Phi(t,t_0) = I + A(t-t_0) + \frac{A^2}{2!}(t-t_0)^2 + \cdots + \frac{A^n}{n!}(t-t_0)^n + \cdots$

Since this serious is obtained as a result of Picard's iteration.

$$X(t) = X(t_0) + \int_{t_0}^{t} A(t')X(t')\,dt'$$

$X_0(t) = X_0(t_0) = I$ — const

$X_1(t) = I + \int_{t_0}^{t} A \cdot I \cdot dt' = I + A(t-t_0)$ 1^{st} iteration — PICARD'S

$X_2(t) = I + \int_{t_0}^{t} A\left(I + A(t'-t_0)\right)dt' = I + A(t-t_0) + \frac{A^2}{2}(t-t_0)^2$ 2^{nd} iter.

$\vdots$

$\therefore \sum_{i=0}^{\infty} A^i \frac{(t-t_0)^i}{i!}$ converges (by fundamental theorem) and is the solution of $\dot{X}(t) = A\,X(t)$ corresponding to the initial condition $X(t_0) = I$.

Call $\boxed{\sum_{i=0}^{\infty} A^i \frac{(t-t_0)^i}{i!} = e^{A(t-t_0)}}$ or $e^{A(t-t_0)} \triangleq \sum_{i=0}^{\infty} \frac{A^i (t-t_0)^i}{i!}$

Lemma: If $A(t)$ and $\int_{t_0}^{t} A(t')\,dt'$ commute (i.e. $A(t)\int_{t_0}^{t} A(t')\,dt' = \left[\int_{t_0}^{t} A(t')\,dt'\right] A(t)$)
then $\Phi(t,t_0) = \sum_{i=0}^{\infty} \left(\int_{t_0}^{t} A(t')\,dt'\right)^i \cdot \frac{1}{i!}$

Proof: It is claimed that $\Phi(t,t_0) = \sum_{i=0}^{\infty} \left(\int_{t_0}^{t} A(t')\,dt'\right)^i \cdot \frac{1}{i!}$

1) Show that the series $\sum_{i=0}^{\infty} \left(\int_{t_0}^{t} A(t')\,dt'\right)^i \frac{1}{i!}$ converges

2) Show that $\sum_{i=0}^{\infty} \left(\int_{t_0}^{t} A(t')\,dt\right)^i \frac{1}{i!}$ is a solution of the $\dot{X}(t) = A(t)X(t)$

$$\frac{d}{dt}\left(\sum_{i=0}^{\infty}\left[\int_{t_0}^{t} A(t')\,dt'\right]^i \frac{1}{i!}\right) = \frac{d}{dt}\Big\{ I + \int_{t_0}^{t} A(t')\,dt' + \frac{1}{2}\left[\int_{t_0}^{t} A(t')\,dt'\right]\left[\int_{t_0}^{t} A(t')\,dt'\right]$$

$$+ \left[\int_{t_0}^{t} A(t')\,dt'\right]\left[\int_{t_0}^{t} A(t')\,dt'\right]\left[\int_{t_0}^{t} A(t')\,dt'\right]\frac{1}{3!} + \cdots \Big\}$$

interchange

$$= A(t) + \frac{1}{2}\left(A(t)\int_{t_0}^{t} A(t')\,dt' + \left(\int_{t_0}^{t} A(t')\,dt'\right)\cdot A(t)\right) + \frac{1}{3!}\Big[A(t)\left(\int_{t_0}^{t} A(t')\,dt'\right)^2 +$$

interchange — interchange from Lemma

$$\left[\int_{t_0}^{t} A(t')\,dt'\right] A(t)\left[\int_{t_0}^{t} A(t')\,dt'\right] + \left[\int_{t_0}^{t} A(t')\,dt'\right]^2 \cdot A(t)\Big) + \cdots$$

8.00

$$= A(t) + A(t)\int_{t_0}^{t} A(t')\,dt' + \frac{1}{2!} A(t)\left[\int_{t_0}^{t} A(t')\,dt'\right]^2 + \dots + \frac{1}{k!} A(t)\left[\int_{t_0}^{t} A(t')\,dt'\right]^k$$

9.00

$$= A(t)\left[I + \int_{t_0}^{t} A(t')\,dt' + \dots + \frac{1}{k!}\left[\int_{t_0}^{t} A(t')\,dt'\right]^k + \dots\right]$$

$$= A(t)\sum_{i=0}^{\infty}\left(\int_{t_0}^{t} A(t')\,dt'\right)^i \frac{1}{i!}$$ → solution olduğu gösterildi

10.00

Define $\displaystyle\sum_{i=0}^{\infty}\left(\int_{t_0}^{t} A(t')\,dt'\right)^i \cdot \frac{1}{i!} = e^{\int_{t_0}^{t} A(t')\,dt'}$ shown that ∃ it converges

11.00

3) $\displaystyle\sum_{i=0}^{\infty}\left(\int_{t_0}^{t} A(t')\,dt'\right)^i \frac{1}{i!}\Big|_{t=t_0} = I$ → initial condition ✓

Example:

12.00

$$A(t) = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix} \qquad \int_{t_0}^{t} A(t')\,dt' = \int_{t_0}^{t}\begin{bmatrix} t' & 1 \\ 0 & t' \end{bmatrix} dt'$$

13.00

$$= \begin{bmatrix} t^2/2 - t_0^2/2 & t - t_0 \\ 0 & t^2/2 - t_0^2/2 \end{bmatrix}$$

14.00

$$A(t)\cdot\int_{t_0}^{t} A(t')\,dt' = \begin{bmatrix} t\left(\frac{t^2}{2} - \frac{t_0^2}{2}\right) & t(t-t_0) + \frac{t^2}{2} - \frac{t_0^2}{2} \\ 0 & t\left(\frac{t^2}{2} - \frac{t_0^2}{2}\right) \end{bmatrix}$$ same

aynı sonucu verirler

15.00

$$\int_{t_0}^{t} A(t')\,dt' \cdot A(t) = \begin{bmatrix} t\left(\frac{t^2}{2} - t_0^2/2\right) & \frac{t^2}{2} - \frac{t_0^2}{2} + t(t-t_0) \\ 0 & t\left(\frac{t^2}{2} - \frac{t_0^2}{2}\right) \end{bmatrix}$$

16.00

$$\therefore\ \Phi(t, t_0) = \sum_{i=0}^{\infty}\begin{bmatrix} t^2/2 - t_0^2/2 & t - t_0 \\ 0 & \frac{t^2}{2} - t_0^2/2 \end{bmatrix}^i \cdot \frac{1}{i!}$$

17.00

önemli **Lemma:** If $A(t)$ is one of the following form then $A(t)$

18.00

commutes with $\int_{t_0}^{t} A(t')\,dt'$

a) $A(t) = A$: constant matrix

19.00

b) $A(t) = \alpha(t).A$ where $\alpha(t)$ is a scaler valued piecewise cont. fn

c) $A(t) = \sum_{i=1}^{k} \alpha_i(t) M_i$ where $\alpha_i(t)$ are piecewise cont. scalar valued functions. M_i's are constant matrices such that $M_i M_j = M_j M_i$ for $i=1,\dots$ $j=1,\dots$

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EYLÜL 1999					
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EKİM 1999					
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Example:

$$A(t) = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix} = \underset{\alpha_1(t)}{t}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M_1} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2} \cdot \underset{\alpha_2(t)}{1}$$

$\underset{I}{M_1} M_2 = M_2 . M_1 \qquad I.M_2 = M_2.I = M_2$

Example: → using linearity

$$\dot{X} = A_1(t)\, X(t) + X(t) . A_2(t) \quad , \quad X(t_0) = X_0$$

Suppose that $\dot{X}(t) = A_1(t)\, X(t) \rightarrow \Phi_1(t, t_0)$

$\dot{X}(t) = A_2(t)\, X(t) \longrightarrow \Phi_2(t, t_0)$

Then

$$X(t) = \Phi_1(t, t_0)\, X_0\, \Phi_2^T(t, t_0)$$

Exercise: Proof that above statement is collect

$\boxed{\text{Properties of } \Phi(t, t_0)}$

1) $\Phi(t, t_0)$ is unique (fundamental theorem)

2) The solution of $\dot{x}(t) = A(t).x(t)$, $x(t_0) = x_0$ is given by $x(t) = \Phi(t, t_0).x_0$

(Proof) $X(t_0) = \Phi(t_0, t_0) X_0 = X_0$,

$$\frac{d}{dt} X(t) = \frac{d}{dt} \Phi(t, t_0)\, X_0 = A(t)\, \Phi(t, t_0) X_0 = A(t) . X(t)$$

3) $\forall\ t_2, t_1, t_0$ we have

$$\Phi(t_2, t_1) = \Phi(t_2, t_0)\, \Phi(t_0, t_1)$$

4) $\Phi(t, t_0)$ is nonsingular $\forall t, \forall t_0$. since it is a fundamental matrix.

Furthermore (3) above indicate that $\Phi(t, t_0)^{-1} = \Phi(t_0, t)$

Proof: From 3 we know that

$I = \Phi(t, t) = \Phi(t, t_0)\, \Phi(t_0, t)$ i.e $t_2 = t_1 = t$

$$\Rightarrow \Phi(t, t_0)^{-1} = \Phi(t_0, t)$$

5) $\Phi(t, t_0) = X(t) . X^{-1}(t_0)$

Proof: $\frac{d}{dt}\left[X(t)\, X^{-1}(t_0) \right] = \dot{X}(t)\, X^{-1}(t_0) = A(t) X(t) X(t_0)$ ∴ RHS satisfies de

$X(t)\, X^{-1}(t_0)|_{t=t_0} = I$

8.00
System

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

9.00
$$y(t) = C(t)\,x(t) + D(t)\,u(t) \qquad (*)$$

Solution of $\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \quad x(t_0) = x_0$ Do we have a unique solu?

10.00
Note that: ① $f(t, x) = A(t)\,x + B(t)\,u(t)$ is a piecewise cont. funct. of time 't' since the matrices

11.00
$A(\cdot)$ & $B(\cdot)$ are piecewise continuous and since we apply an input $u(\cdot)$ which is piecewise continuous

12.00
② $f(t,x)$ satisfies a Lipschitz condition on x since

$$\| A(t)\,x + \cancel{B(t)\,u(t)} - A(t)\,\bar{x} - \cancel{B(t)\,u(t)} \| = \| A(t)(x-\bar{x}) \|$$

13.00
$$\leq \| A(t) \| \, \| x - \bar{x} \| = k(t) \cdot \| x - \bar{x} \| \quad (k(t) \geq 0)$$

$\therefore$ (*) has a unique solution

14.00

③ Calculation of the solution by the method of variation of Parameters: variational equation

15.00
$x(t) = \phi(t,t_0)\,x_0$

$$\dot{x}(t) = A(t)\,x(t) \longrightarrow \Phi(t, t_0) \cdot x_0 \qquad x(t_0) = x_0$$

16.00
$$x(t) = \Phi(t,t_0) \cdot \xi(t) \qquad \xi(t): \text{unknown timevarying vector}$$

17.00
$$\dot{x}(t) = \dot{\Phi}(t,t_0)\,\xi(t) + \Phi(t,t_0) \cdot \dot{\xi}(t) = A(t) \cdot \Phi(t,t_0)\,\xi(t) + B(t) \cdot u(t)$$

$$\Rightarrow A(t) \cdot \Phi(t,t_0)\,\xi(t) + \Phi(t,t_0) \cdot \dot{\xi}(t) = A(t)\,\Phi(t,t_0) \cdot \xi(t) + B(t)\,u(t)$$

18.00
$$\Rightarrow \Phi(t,t_0)\,\dot{\xi}(t) = B(t)\,u(t) \xrightarrow{\text{tersini al}} \dot{\xi}(t) = \phi(t_0,t)\,B(t)\,u(t)$$

19.00
$$\int_{\xi(t_0)}^{\xi(t)} d\xi(t') = \int \Phi(t_0,t')\,B(t')\,u(t')\,dt'$$

$$\xi(t) - \xi(t_0) = \int_{t_0}^{t} \Phi(t_0,t')\,B(t')\,u(t')\,dt'$$

$$\xi(t) = \xi(t_0) + \int_{t_0}^{t} \Phi(t_0, t') B(t') u(t')\, dt' = x_0 + \int_{t_0}^{t} \phi(t_0, t') B(t') u(t')\, dt'$$

Note that: $x(t)\Big|_{t_0} = \Phi(t, t_0)\, \xi(t)\Big|_{t_0} \Rightarrow x(t_0) = \xi(t_0) = x_0$ $\quad x(t) = \phi(t, t_0)\xi$

$\therefore\ x(t) = \Phi(t, t_0) x_0 + \Phi(t, t_0) \int_{t_0}^{t} \Phi(t_0, t') B(t') u(t')\, dt'$

$$\circledast \quad \boxed{x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, t') B(t') u(t')\, dt'}$$

Theorem: $A = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$ is a linear dynamical system

Proof:

T : R^+

U : Set of all piecewise cont. functions mapping $T \to R^r$ ($\mathbb{C}^r$) $U = R^r$ complex Range

Y : Set of all " " " " $T \to R^m$ ($Y = R^m$)

$\Sigma = R^n$ $\quad x(t)$: State at time 't'

$x(t) = s(t, t_0, x_0, u) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, t') B(t') \cdot u(t')\, dt'$

read out map $y(t) = C(t) x(t) + D(t) u(t)$

State transition axiom:

Note that in $\circledast$ the part of the input $u(t')$ where $t_0 \le t' \le t$ is used

$s(t, t_0, x_0, u) = s(t, t_0, x_0, \tilde{u})$

if $u(t') = \tilde{u}(t')$ $\quad \forall\, t' \in [t_0, t]$

Piecewise cont

cont

Piecewise > cont

eşitlik var

Semi group axiom:

$$s(t_2, t_1, \underbrace{s(t_1, t_0, x_0, u)}_{\Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, t') B(t') u(t')\, dt'}, u) = s(t_2, t_0, x_0, u)$$

$$x_2(t) = \Phi(t_2, t_1)\left[\Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, t') B(t') u(t')\, dt'\right] + \int_{t_1}^{t_2} \Phi(t_2, t') B(t') u(t')$$

$$= \Phi(t_2, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_2, t') B(t') u(t')\, dt' + \int_{t_1}^{t_2} \Phi(t_2, t') B(t') u(t')\, dt'$$

$$= \Phi(t_2, t_0) x_0 + \int_{t_0}^{t_2} \Phi(t_2, t') B(t') u(t')\, dt' = \text{RHS}.$$

$\therefore\ R = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$ is a dynamical system.

20 PAZARTESİ 21 SALI 22 ÇARŞAMBA 23 PERŞEMBE 24 CUMA 25 CUMARTESİ

8.00 To prove linearity : Note that U, Σ and Y are linear spaces over the field R.

9.00 Response function

$$y(t) = g(t, t_0, x_0, u) = C(t)\,\Phi(t,t_0)\,x_0 + C(t)\int_{t_0}^{t}\Phi(t,t')B(t')u(t')dt' + D(t)\,u(t)$$

10.00

11.00 $\therefore$ Let ① $(x_0, u(\cdot))$, ② $(\bar{x}_0, \bar{u}(\cdot))$ $\rightarrow$ ③ $(\alpha x_0 + \beta \bar{x}_0), (\alpha u + \beta \bar{u})$ For linearity

12.00

$$① \rightarrow y(t) = \underbrace{C(t)\,\Phi(t,t_0)\,x_0}_{\text{zero input response}} + \underbrace{C(t)\int_{t_0}^{t}\Phi(t,t')B(t')u(t')dt' + D(t)\,u(t)}_{\text{zero state response}}$$

13.00
$$② \rightarrow \bar{y}(t) = C(t)\,\Phi(t,t_0)\,\bar{x}_0 + C(t)\int_{t_0}^{t}\Phi(t,t')B(t')\bar{u}(t')dt' + D(t)\,\bar{u}(t)$$

$$③ \rightarrow \tilde{y}(t) = C(t)\Phi(t,t_0)\left[\alpha x_0 + \beta \bar{x}_0\right] + C(t)\int_{t_0}^{t}\Phi(t,t')B(t')\left[\alpha u(t') + \beta \bar{u}(t')\right]dt' + D(t)\left[\alpha u(t) + \beta \bar{u}(t)\right]$$

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14.00

Note that ③ = α · ① + β ② so system is linear.

15.00

16.00

17.00

18.00

19.00

Dünya Alzheimer Günü

AĞUSTOS 1999					
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EYLÜL 1999					
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EKİM 1999					
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$$\dot{x} = f(x,t,u) \longrightarrow \left.\begin{array}{l} x(t_0) = x_0 \\ u(t) : \text{input} \end{array}\right\} \Rightarrow x(t)$$

$$u(t) \longrightarrow \left.\begin{array}{l} u(t) + \delta u(t) \\ x(t_0) = x_0 \end{array}\right\} \Rightarrow \tilde{x}(t) = x(t) + \delta x(t)$$

$$\delta\dot{x} = A(t)\,\delta x + B(t)\,\delta u$$

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{u(t),\,x(t)} \qquad B(t) = \left.\frac{\partial f}{\partial u}\right|_{u(t),\,x(t)}$$

$$\begin{tikzcd}
\dot{x} = A(t)\,x(t) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \dot{\xi} = -A^*(t)\,\xi \arrow[d, leftrightarrow] \\
\Phi(t,t_0) \arrow[dr] & \Psi(t,t_0) \arrow[d] \\
& \Psi^*(t,t_0) = \Phi(t_0,t)
\end{tikzcd}$$

Optimization Problem:

$$\dot{x}(t) = f(t,x,u), \quad x(t_0) = x_0, \quad t_1 : \text{final time: fixed}$$

Aim is $\max \varphi(x(t_1))$

$$\underset{t_0,\,x_0}{|}\text{―――――――――}\underset{t_1}{|} \quad \varphi(x(t_1)) = -\|x(t_1) - a\|^2$$

$u(t) \rightarrow x(t)$ change the input to $u(t) + \delta u(t)$. Then the corresponding solution will be $\tilde{x} = x(t) + \delta x(t)$ where $\delta x(t)$ satisfies the d.e.

$$\delta\dot{x}(t) \cong A(t)\,\delta x(t) + B(t)\,\delta u(t) \qquad \delta x(t_0) = 0$$

where $A(t) = \left.\frac{\partial f}{\partial x}\right|_{x(t),\,u(t)}$ $\qquad B(t) = \left.\frac{\partial f}{\partial u}\right|_{x(t),\,u(t)}$

Objective function for the new input

$$\varphi(\tilde{x}(t_1)) = \varphi(\underbrace{x(t_1)}_{x_1} + \delta x(t_1))$$

$$\varphi(\tilde{x}(t_1)) = \varphi(x_1) + \left.\frac{\partial \varphi}{\partial x}\right|_{x = x_1} \delta x(t_1) + \ldots \quad \text{if } \delta \text{ is small}$$

$$\varphi(\tilde{x}(t_1)) - \varphi(x_1) \cong \left.\frac{\partial \varphi}{\partial x}\right|_{x = x_1} \delta x(t_1) \overset{?}{\geq} 0$$

Find a $\delta u(t)$ for good result.

$\max \psi(X(t_1)) = \varphi(X_1(t_1), X_2(t_1), \ldots X_n(t_1))$; $\varphi : R^n \to R$

$$\delta X(t_1) = \int_{t_0}^{t_1} \Phi(t_1, t') B(t') \delta u(t') dt'$$

$$\psi(\tilde{X}(t_1)) - \psi(X(t_1)) \approx \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix}^T}_{g} \cdot \int_{t_0}^{t} \Phi(t_1, t') B(t') \delta u(t') dt'$$

__Problem:__ Find δu such that $\psi(\tilde{X}(t_1)) - \psi(X(t_1)) > 0$

let $\delta u(t') = (B^T(t') \Phi^T(t_1, t') g) \cdot \alpha$ → like a steep. descent method

Then $\int_{t_0}^{t_1} \| B^T(t') \underbrace{\Phi^T(t_1, t') g} \|^2 dt' > 0$

$\dot{X} = A(t) X(t)$ $\quad \dot{\xi} = -A^*(t) \xi(t)$

$\Phi(t_0, t) = \psi(t, t_0)$

$\psi(t', t_1) g$: Solution of the adjoint equ.

Note that $\Phi^T(t_1, t') g = \psi(t_1, t') g$. So this equation (expression) in the solution of the adjoint equation starting from the final condition "g" backward

__Example__

$\dot{X}(t) = -X(t) + u(t)$ $\quad t=0, \; t_1 = 1 \quad X(t_0) = X_0 = 10 \quad \max \{-X(1)\}^2 = ?$

$u(t) = 0 \quad \dot{X}(t) = -X(t) \quad X(t) = 10 e^{-t} \quad X(1) = 10/e \quad \varphi(X(1)) = -(\frac{10}{e})^2$

$u(t) \to u(t) + \delta u(t) \to \tilde{X}(t) = X(t) + \delta X(t) \quad \delta\dot{X} = -\delta X + \delta u \quad \delta X(0) = 0$

$\varphi(\tilde{X}(1)) \approx -X(1)^2 + (-2X)_{X(t_1)} \cdot \delta X(t_1) = -(\frac{10}{e})^2 - \underbrace{2 \cdot \frac{10}{e} \delta X(t_1)}_{\geq 0}$

$\delta X(1) = \int_0^1 e^{-(1-t')} \delta u(t') dt' \Rightarrow -2 \frac{10}{e} \int_0^1 e^{-(1-t')} \delta u(t') dt' > 0$

$\Rightarrow \delta u(t') = [-2 \frac{10}{e} e^{-(1-t')}] \alpha$

$= [-\frac{20}{e} e^{-1} e^{t'}] \alpha$

$B = \frac{dx}{du} \Big| = \frac{d(-x(1) + u}{du} = 1$

$\delta\dot{x} = A(t)\delta x(t) + B(t)\delta u(t)$ $\quad A = \frac{dx}{dx} \Big| = \frac{d(-x(t) + u(t))}{dx} = -1$

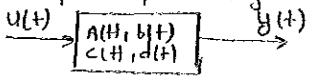
$\delta\dot{x} = -\delta x + \delta u$

8.00

Impulse Response:

Case 1: Single input single output

9.00



Inp.^{n} Applied zero initial cond. for impulse resp.

$\delta(t)$: Unit impulse

10.00

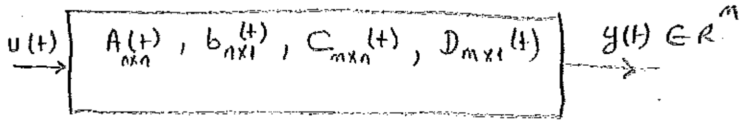
$$y(t) = c(t)\int_{-\infty}^{t} \Phi(t,t')\, b(t')\, \delta(t'-\tau)\,dt' + d(t)\cdot\delta(t-\tau)$$

11.00

$$h(t,\tau) = c(t)\cdot\Phi(t,\tau)b(\tau) + d(t)\,\delta(t-\tau)$$

Case 2: Single input multi-output

12.00



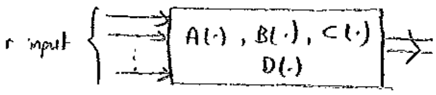
13.00

$$H(t,\tau) = c(t)\,\Phi(t,\tau)b(\tau) + D(t)\cdot\delta(t-\tau)$$

14.00

Case 3: Multi input / multi output

15.00



16.00

$$\left.\begin{aligned} u_1(t) &= \delta(t-\tau) \\ u_2(t) &= \dots = u_r(t) = 0 \end{aligned}\right\}$$ $y_1(t)$: first column of the impulse response matrix

let $B = [b_1 b_2 \dots b_r]$ $D(t) = [d_1\ d_2 \dots d_r]$

17.00

then

$$y_1(t) = C(t)\,\Phi(t,\tau)\, b_1(\tau) + d_1(\tau)\,\delta(t-\tau)$$

18.00

Apply $u(t)$ st $u_1(t) = u_3(t) = \dots = u_r(t) = 0$

$$u_2(t) = \delta(t-\tau)$$

19.00

$$y_2(t,\tau) = c(t)\,\Phi(t,\tau)\, b_2(\tau) + d_2(\tau)\,\delta(t-\tau)$$

$\vdots$

Define the impulse response matrix as

$$H(t,\tau) = [\,y_1(t,\tau)\quad y_2(t,\tau)\ \dots\,] = C(t)\,\Phi(t,\tau)B(\tau) + D(\tau)\,\delta(t-\tau)$$

$R = [A, B, C, D]$ Representation

$A \to n \times n$ constant matrix

$B \to n \times r$ " "

$C \to m \times n$ " "

$D \to m \times r$ " "

Homogeneous equation

$\dot{x} = Ax$

$\dot{X} = AX \to \Phi(t, t_0)$

$\hookrightarrow X(t_0) = I$

$$\int_{X(t_0)}^{X(t)} \dot{X}(t') = \int_{t_0}^{t} A\, X(t')\, dt'$$

$$X(t) = X(t_0) + \int_{t_0}^{t} A\, X(t')\, dt'$$

From Picard $\Rightarrow$

$$\begin{cases} X_0(t) = I \\ X_1(t) = I + \int_{t_0}^{t} A \cdot I\, dt' = I + A(t - t_0) \\ X_2(t) = I + \int_{t_0}^{t} A\,(I + A(t' - t_0))\,dt' = I + A(t - t_0) + A^2 \left(\frac{t - t_0}{2}\right)^2 \end{cases}$$

$\vdots$

$$X_k(t) = I + A(t - t_0) + \frac{A^2}{2}(t - t_0)^2 + \cdots + \frac{A^k (t - t_0)^k}{k!}$$

$$X(t) = \sum_{i=0}^{\infty} \frac{A^i (t - t_0)^i}{i!} = \Phi(t, t_0)$$

$$\boxed{\Phi(t, t_0) = \sum_{i=0}^{\infty} A^i \frac{(t - t_0)^i}{i!} \triangleq e^{A(t - t_0)}}$$

$$e^{A(t - t_0)} \triangleq \sum_{i=0}^{\infty} A^i \frac{(t - t_0)^i}{i!}$$

Example: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad e^{At} = ?$

Solution: Use the definition $\quad e^{At} = \Phi(t, 0)$

$$A^i = \begin{bmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{bmatrix}$$

$$e^{At} = \sum_{i=0}^{\infty} (t)^i \begin{bmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{bmatrix} \frac{1}{i!} = \begin{bmatrix} \sum_{i=0}^{\infty} \frac{t^i \lambda_1^i}{i!} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{t^i \lambda_2^i}{i!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{\infty} \frac{t^i \lambda_1^i}{i!} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{t^i \lambda_2^i}{i!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Second way of calculating $e^{At} = \Phi(t,0)$

$$\dot{X} = AX$$

$$sX(s) - I = AX(s)$$

$$(sI - A)\,X(s) = I$$

$$X(s) = (sI - A)^{-1}$$

$$\Phi(t,0) = e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} s-\lambda_1 & 0 \\ 0 & s-\lambda_2 \end{bmatrix}^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-\lambda_1} & 0 \\ 0 & \frac{1}{s-\lambda_2} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Problem 2: Find $\Phi(t,t_0)$

1.) Note that $\Phi(t,t_0) = \sum_{i=0}^{\infty} A^i \frac{(t-t_0)}{i!}$

and $\Phi(t,0) = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!}$

∴ By comparing above expression we have

$$\Phi(t,t_0) = \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix}$$

2) $\Phi(t,t_0) = \Phi(t,0)\,\Phi(0,t_0) = \Phi(t,0)\cdot\Phi^{-1}(t_0,0)$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t_0} & 0 \\ 0 & e^{\lambda_2 t_0} \end{bmatrix}^{-1} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} e^{-\lambda_1 t_0} & 0 \\ 0 & e^{-\lambda_2 t_0} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_0 \end{bmatrix} \qquad \text{Find } e^{At} = \Phi(t,0)$$

Claim: The solution is $\Phi(t,0) = e^{At} = \begin{bmatrix} e^{\lambda_0 t} & te^{\lambda_0 t} & \cdots & \frac{t^{n-1}}{(n-1)!} e^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} & \cdots & \frac{t^{n-2}}{(n-2)!}, e^{\lambda_0 t} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & e^{\lambda_0 t} \end{bmatrix}$

Proof of the claim:

1) $\Phi(0,0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} = e^{At}\Big|_{t=0}$ It satisfies the initial condition

2) $$\frac{d}{dt}\left[\Phi(t,0)\right] = \underbrace{\begin{bmatrix} \lambda_0 e^{\lambda_0 t} & e^{\lambda_0 t} + \lambda_0 t e^{\lambda_0 t} & \frac{t^{n-2}}{(n-2)!} e^{\lambda_0 t} + \lambda_0 \frac{t^{n-1}}{(n-1)!} e^{\lambda_0 t} \\ 0 & \lambda_0 e^{\lambda_0 t} & \frac{t^{n-3}}{(n-3)!} e^{\lambda_0 t} + \lambda_0 \frac{t^{n-2}}{(n-2)!} e^{\lambda_0 t} \\ \vdots & \vdots & \\ 0 & 0 & \lambda_0 e^{\lambda_0 t} \end{bmatrix}}_{B}$$

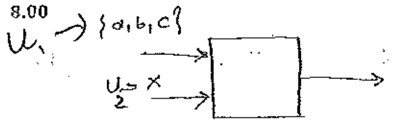
$$= \begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & & & \lambda_0 \end{bmatrix} \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} = \begin{bmatrix} \lambda_0 e^{\lambda_0 t} & \lambda_0 t e^{\lambda_0 t} + e^{\lambda_0 t} & \cdots & \lambda_0 \frac{t^{n-1}}{(n-1)!} e^{\lambda_0 t} + \frac{t^{n-2}}{(n-2)!} e^{\lambda_0 t} \\ 0 & \lambda_0 e^{\lambda_0 t} & \cdots & \lambda_0 e^{\lambda_0 t} \frac{t^{n-2}}{(n-2)!} \\ \vdots & & & \\ 0 & 0 & & \lambda_0 e^{\lambda_0 t} \end{bmatrix}$$

$\therefore$ It satisfies the d.e.

$$\begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & 0 \cdots & 0 \\ \vdots & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \lambda_0 \end{bmatrix} \begin{bmatrix} e^{\lambda_0 t} & t e^{\lambda_0 t} & \frac{t^2}{2} e^{\lambda_0 t} & \cdots & \frac{t^{n-1}}{(n-1)!} e^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Example

8.00



$x_k \in R$

$y_k = 0.5 x_{k-1} + x_k +$ total of 'a''s in the first input

9.00

$= 0.5 u_{k-1} + u_k +$ "

$T : \mathbb{Z}$

10.00

$U = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{array}{l} u_1\text{: string of letters } a,b,c \\ u_2\text{: Sequence of real numbers} \end{array} \right\}$ $\quad U = \left\{ \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \middle| \begin{array}{l} u_1(k) \in \{a,b,c\} \\ u_2(k) \in R \end{array} \right\}$

11.00

y: String of real numbers , $Y = R$

12.00

Σ : $\quad x_1(k) =$ Number of a's in the input string starting $-\infty$ upto k

$x_2(k) = u_2(k-1)$ (k is excluded)

13.00

$$S\left(k, k_0, \begin{matrix} x_{10} \\ x_{20} \end{matrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{[k_0,k]}\right) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \left(\begin{array}{l} x_{10} + \text{Number of 'a''s in } u_1[\cdot] \text{ on the interval } [k_0, k) \\ u_2(k-1) \quad \text{if } k-1 > k_0 \\ x_2(k_0) \quad k = k_0 \end{array} \right.$$

14.00

15.00

$$r(k, x(k), u(k)) = 0.5\, x_2(k) + u_2(k) + x_1(k) + \left\{ \begin{array}{ll} 1 & \text{if } u_1(k) = a \\ 0 & \text{otherwise} \end{array} \right\}$$

Not linear, but time invar. system

16.00

$\sin, \cos, e^{\alpha t}, u(t), r(t) \overset{\text{Laplace}}{=} ?$

17.00

18.00

19.00

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = \mathcal{L}^{-1}(sI-A)^{-1}$$

Definition: $d(s) = \det(sI-A)$ is called the characteristic polynomial of the matrix A. $\det(sI-A)=0$ is the characteristic equation roots of this equation are the eigenvalues of the matrix A. If A is an $n \times n$ matrix then $\det(sI-A) = |sI-A| \sim s^n + a_1 s^{n-1} + \dots + a_n$

$(sI-A)^{-1} = ?$ Assume that $d(s)$ is known.

$$(sI-A)^{-1} = \frac{1}{d(s)} \cdot \mathrm{adj}(sI-A) = \frac{s^{n-1}B_0 + s^{n-2}B_1 + \dots + B_{n-1}}{s^n + a_1 s^{n-1} + \dots + a_n} \quad B_i : n \times n$$

$$\underset{(sI-A)}{(s^n + a_1 s^{n-1} + \dots + a_n)\wedge}(sI-A)^{-1} = \underset{(sI-A)}{\wedge}(s^{n-1}B_0 + s^{n-2}B_1 + \dots + B_{n-1})$$

$$(s^n + a_1 s^{n-1} + \dots + a_n) I = s^n B_0 + s^{n-1} B_1 + \dots + s B_{n-1}$$
$$- s^{n-1} A B_0 + \dots - s A B_{n-2} - A B_{n-1}$$

$$s^n I + a_1 s^{n-1} I + \dots + a_n I = s^n B_0 + s^{n-1}(B_1 - AB_0) + \dots + s(B_{n-1} - AB_{n-2}) - AB_{n-1}$$

$I = B_0$

$a_1 I = B_1 - A \Rightarrow B_1 = A + a_1 I$

$a_2 I = B_2 - AB_1 \Rightarrow B_2 = A(A + a_1 I) + a_2 I = A^2 + a_1 A + a_2 I$

$a_3 I = B_3 - AB_2 \Rightarrow B_3 = A^3 + a_1 A^2 + a_2 A + a_3 I$

$\vdots$

$a_{n-1} I = B_{n-2} - AB_{n-3} \Rightarrow B_{n-1} = A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_{n-1} I$

$a_n I = -A B_{n-1}$

$a_n I + A B_{n-1} = [0]$

$a_n I + A^n + a_1 A^{n-1} + \dots + a_{n-1} A = 0$

$\rightarrow \boxed{A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = [0]} \Rightarrow d(s)$

Theorem: (Cayley Hamilton) Let A be an $n \times n$ matrix. Let $d(s) = s^n + a_1 s^{n-1} + \dots + a_n$ be its characteristic polynomial then $d(A) \triangleq A^n + a_1 A^{n-1} + \dots + a_n I = [0]$

Every matrix satisfies its characteristic equation